# Special values of L-functions and false Tate curve extensions II

Thanasis Bouganis

February 3, 2008

**Abstract**

In this paper we show how one can combine the $p$-adic Rankin-Selberg product construction of Hida with freeness results of Hecke modules of Wiles to establish interesting congruences between special values of L-functions. These congruences is a part of some deep conjectural congruences that follow from the work of Kato on the non-commutative Iwasawa theory of the false Tate curve extension.

## 1 Introduction

Let $E$ be an elliptic curve defined over $\mathbb{Q}$ and $p$ a rational prime. In the classical setting of cyclotomic Iwasawa theory for elliptic curves one is concerned with the study of the twists of the elliptic curve by finite order character that factor through the cyclotomic $\mathbb{Z}_p$ extension $\mathbb{Q}_{cyc} \subset \cup_{n\geq 0}\mathbb{Q}(\mu_{p^n})$, where $\mu_{p^n}$ is the group of the $p^n$-th roots of unity. The aim of the theory is to obtain a link between the analytically defined $L$ functions attached to $E$, and its twists, and the arithmetic properties of the elliptic curve over the cyclotomic tower. The cyclotomic Main Conjecture for elliptic curves gives to this conjectural link a very precise form. We note that much has already been proven towards this Main Conjecture by Kato [19], and Skinner and Urban have announced a complete proof for semi-stable $E$, subject to proving certain results about the Galois representations attached to automorphic forms.

One of the key ingredients of the above Main Conjecture are the $p$-adic $L$ functions. These are usually realized as $p$-adic measures over Galois groups, which, when evaluated at finite order characters, interpolate canonically modified values of the $L$ function. Their construction usually involves two steps. The first one is to find proper transcendental numbers, usually called periods, such that the ratio of the $L$ values over these periods gives an algebraic number. The second step is to prove that these values, or a slight modification of them, have the desired interpolation and integrality properties.

Lately there has been great interest in extending the classical Iwasawa theory to a non abelian setting, that is to replace the $\mathbb{Z}_p$ extension by more general $p$-adic Lie extensions whose Galois group is non-abelian. In fact in [5] a precise analogue of the Main Conjecture in this non abelian setting for a large family of $p$-adic Lie groups has been stated.

One of the extensions that is of particular interest is the so called "false Tate curve" extensions. That is extensions of the form, $\mathbb{Q}_{FT} := \cup_{n\geq 0}\mathbb{Q}(\mu_{p^n}, \sqrt[p^n]{m})$ for some $p$-power free integer $m > 1$. Note that the Galois group is the semi-direct product $\mathbb{Z}_p \ltimes \mathbb{Z}_p^\times$. There is a conjectural theory for $p$-adic $L$ functions that should exist in this setting. In a work with V.Dokchitser [4] we have addressed the first of the above mentioned two steps, that is algebraicity of the critical values of the $L$ functions involved.

In order to make things more explicit let us fix some more notation. We write $E$ for an elliptic curve defined over $\mathbb{Q}$ and $N_E$ for its conductor. As we already mentioned we consider the extensions $\mathbb{Q}_{FT,n} := \mathbb{Q}(\mu_{p^n}, \sqrt[p^n]{m})$ and $\mathbb{Q}_{FT} = \cup_{n\geq 0}\mathbb{Q}_{FT,n}$. We write $\rho$ for an Artin representation that factors through $\mathbb{Q}_{FT}$ and $N_\rho$ for its conductor. Let us also write $L(E,\rho,s)$ for the $L$ function attached to $E$ twisted by $\rho$. We consider the value of $L(E,\rho,s)$ at the critical point $s = 1$. The fact that the Artin representations $\rho$ factor through the false Tate curve allowed us to establish the analyticity of $L(E,\rho,s)$ at $s = 1$ and then our main result in [4] is concerned with the algebraic properties of these values. Let us write $\Omega_\pm(E)$ for the Néron periods attached to the elliptic curve $E$. Then we have shown that

$$\frac{L(E,\rho,1)}{\Omega_+(E)^{dim(\rho^+)}\Omega_-(E)^{dim(\rho^-)}} \in \overline{\mathbb{Q}}.$$

for all Artin representations $\rho$ that factor through $\mathbb{Q}_{FT}$. Actually we did more. Namely, involving also the period that should correspond to the "Artin motive" $M(\rho)$ attached to $\rho$ we established the period conjecture of Deligne that gives a precise description of the number field where this value lies.

Let us now move to the second step that we mentioned above, that is the $p$-adic properties of these values. From now on we will assume that the elliptic curve has good ordinary reduction at $p$. We start by stating a conjectural congruence between these $L$ values for different Artin representations. We define the quantity $R(\rho)$ as

$$R(\rho) := e_p(\rho)u^{-v_p(N_\rho)}\frac{P_p(\hat{\rho},u^{-1})}{P_p(\rho,w^{-1})}\cdot\frac{L_{\{p,q|m\}}(E,\rho,1)}{\Omega_+(E)^{dim(\rho^+)}\Omega_-(E)^{dim(\rho^-)}}$$

where $e_p(\rho)$ is a local epsilon factor of $\rho$ suitably normalized, $P_p(\rho,X)$ is the usual characteristic polynomial associated to $\rho$ at $p$ and $u, w$ are $p$-adic numbers defined by,

$$1 - a_pX + pX^2 = (1-uX)(1-wX), \quad u \in \mathbb{Z}_p^\times \;\; and \;\; p + 1 - a_p = \#E_p(\mathbb{F}_p)$$

Here $\hat{\rho}$ is the dual representation but in our false Tate curve setting it is easy to see that $\hat{\rho} \cong \rho$. Finally the subscript $\{p,q|m\}$ means that we have removed the Euler factors at these primes. Then we state,

**Conjecture:** For each $n \geq 1$, let $\chi_n$ be a character of $Gal(\mathbb{Q}_{FT,n}/\mathbb{Q}(\mu_{p^n}))$ of exact order $p^n$. Write $\rho_n$ for the induced representation of $\chi_n$ to $Gal(\mathbb{Q}_{FT,n}/\mathbb{Q})$ and $\sigma_n$ for the representation induced to $Gal(\mathbb{Q}_{FT,n}/\mathbb{Q})$ from the trivial one over $\mathbb{Q}(\mu_{p^n})$. Then, the values $R(\rho_n)$ and $R(\sigma_n)$ are $p$-adically integral and satisfy

$$|R(\rho_n) - R(\sigma_n)|_p < 1$$

or more generally

$$|R(\rho_n \otimes \psi) - R(\sigma_n \otimes \psi)|_p < 1$$

where $\psi$ is a finite order character of $Gal(\mathbb{Q}^{cyc}/\mathbb{Q})$ and $|\cdot|_p$ normalized as $|p|_p = p^{-1}$.

Let us comment a little bit more on this conjecture and its connection to non commutative Iwasawa theory. The definition of the quantity $R(\rho)$ describes the interpolation properties that the conjectural, as in [5], non-abelian $p$-adic $L$-function should satisfy. Indeed the authors in [5] have conjectured the existence of an element in the $K_1$ of the Iwasawa algebra associated to this extension that interpolates suitably modified, as above, values of $L(E, \rho, 1)$ and plays the role of the non-abelian $p$-adic $L$ function in their theory. Note that the representations $\rho_n$ and $\sigma_n$ are defined over $\mathbb{Q}$ and are congruent modulo $p$ that is if we consider their reduction modulo $p$ then their semi-simplifications are isomorphic. Hence the existence of the non-abelian $p$-adic $L$ function would imply that its values should be also $p$-adically close.

There is almost nothing known concerning the construction of this object for a general $p$-adic Lie extension. However in the setting that we are interested in, the false Tate curve extension, Kato in [18] has related the existence of this non-abelian object with congruences between classical abelian $p$-adic $L$ functions over various fields of the extension. We take some time to explain this as it will help us motivate the results that appear in this paper. Let $G$ be the Galois group of the false Tate curve extension and $\Lambda(G) = \mathbb{Z}_p[[G]]$ the Iwasawa algebra of $G$. We set $U^{(n)} := ker(\mathbb{Z}_p^\times \to (\mathbb{Z}/p^n\mathbb{Z})^\times)$. The main result of Kato in [18] is the construction of an injective homomorphism

$$\theta_G : K_1(\Lambda(G)) \to \prod_{n \geq 0} \mathbb{Z}_p[[U^{(n)}]]^\times$$

and the explicit description of the image. In order to make this last statement a little bit more precise we write, for $n \geq m \geq 0$, $N_{m,n} : \mathbb{Z}_p[[U^{(m)}]] \to \mathbb{Z}_p[[U^{(n)}]]$ for the canonical norm map, $\phi$ be the ring homomorphism $\mathbb{Z}_p[[\mathbb{Z}_p^\times]] \to \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ induced by the rising to the power $p$ map on $\mathbb{Z}_p^\times$. Then the result of Kato says that $\theta_G(K_1(\Lambda(G))) = (a_n)_{n\geq 0}$ with

$$\prod_{0<i\leq n} N_{i,n}(c_i)^{p^i} \equiv 1 \mod p^{2n}$$

with $c_n = b_n\phi(b_{n-1})^{-1}$ and $b_n = a_n N_{0,n}(a_0)^{-1}$. The elements $a_n$ have an arithmetic meaning, they are abelian $p$-adic $L$ functions. More precisely if we write $\rho_n$ for the Artin representation of $G$ induced from a character of $p^n$ order of the Galois group $Gal(\mathbb{Q}(\mu_{p^n}, \sqrt[p^n]{m})/\mathbb{Q}(\mu_{p^n}))$, then the elements $a_n$ are the abelian $p$-adic $L$-functions interpolating the values $L(E \otimes \rho_n \otimes \chi, 1)$, for $\chi$ Dirichlet characters of the cyclotomic extension of $\mathbb{Q}$.

The conjectural congruences that we have written above correspond to the case of $n = 1$ of Kato's congruences after evaluating the abelian $p$-adic $L$ functions at the character $\psi$. There is computational support for these conjectures; initially by Balister [1] and much more vastly by the Dokchitser brothers [11]. In the first part of this work [3] we have showed the existence of the abelian $p$-adic $L$-functions $a_n$ appeared in Katos's congruences and proved the above conjectural congruences up to an issue of periods. Namely there we have used not the motivic periods that are stated in the congruences but automorphic periods, the so called Eichler-Shimura-Harder periods, that appear quite natural in the so called modular symbol construction. There we came across to a rather deep problem, namely the relation of these automorphic periods as one use the functorial properties of the $L$-functions and especially base-change. We say a little bit more on this at the last section of this paper. Finally we note that in [7] an inductive argument was used to show how these congruences (for $n = 1$) can provide congruences for $n > 1$ in the form conjectured by Kato but unfortunately not modulo the right $p$ power.

Our aim in this paper is to tackle the conjectural congruences insisting on getting the right motivic periods. We achieve that for the case where $p = 3$ but we also discuss possible extensions for the case of $p > 3$. We need to impose some further conditions on $E$, other of technical nature which we believe can be removed and other that seem important. Namely from now on we assume that (a) The curve $E$ is semi-stable and if we consider the minimal discriminant $\Delta_E = \prod_{q|N_E} q^{i_q}$ then $p$ does not divide $i_q$ for all $q$. Note that the last condition means that the conductor of $E$ is equal to the Artin conductor of the mod $p$ representation obtained by $E$. (b) We assume that $m$ that appear in the false Tate extension is power free with $(m, N_E) = (m, p) = 1$ and, (c) a rather important assumption, that $E$ has no rational subgroup of order $p$, that is the associated modulo $p$ representation is irreducible. Finally we mention here that as our aim here is to address the issue of motivic versus automorphic periods we focus on proving the above conjectures for $\psi = 1$. However we lay all important constructions so that everything can be extend to the case $\psi$ being not trivial.

Our proof can be divided into two parts. Let us write $f \in S_2(\Gamma_0(N_E); \mathbb{Q})$ for the rational newform that we can associate to $E$. In the first part we rely on the work of Hida of the construction of a p-adic Rankin-Selberg product initiated in [13] and generalized in [14]. We can associate a newform $g$ of weight one to the Artin representation $\rho$ and an Eisenstein series $\mathcal{E}$ of weight one with $\sigma$. Using them, we construct $p$-adic measures $d\mu_{f,g}$ and $d\mu_{f,\mathcal{E}}$ over $\mathbb{Z}_p^\times$ that are congruent modulo $p$, in the sense that their values at every finite character of $\mathbb{Z}_p^\times$ are congruent. These measures interpolate, $p$-adically, twists of the critical values of the Rankin-Selberg products $D(f, g, s)$ and $D(f, \mathcal{E}, s)$ by finite order characters. Evaluating these measures at the trivial character we get a first form of congruences between $D(f, g, 1)$ and $D(f, \mathcal{E}, 1)$. Under the semi-stable assumption we can easily relate the Rankin-Selberg product to the twists of the elliptic curve $E$.

However we do not yet get the congruences stated in the theorem above. We need to work further two things. First, in order to establish the congruences between the measures above, we had to clear a denominator $c(f, m)$ that depends solely on $f$ and $m$. Hence we get congruences after multiplying with this constant $c(f, m)$. Second, the periods that we use to get the rationality of the Rankin-Selberg product are closely related to the Petersson inner product $< f, f >$. These periods may not be equal to our periods $\Omega_+(E)$ and $\Omega_-(E)$ up to a $p$-adic unit. These two problems are related. That is, the reason that the denominator $c(f, m)$ appears in our $p$-adic interpolation is the fact that the Petersson inner product is not the proper automorphic period in order to get $p$-adically integral ratios of the form $\frac{L-values}{aut.\ periods}$.

In the second part we show, under the assumptions of the theorem, that indeed this is the case. This part relies heavily on the work of Wiles. We make use of two of his important results in [28]. The first one is an extension of a theorem of Mazur [24] on the freeness, over a completed Hecke algebra, of the first cohomology group of modular curves after localizing it at a proper maximal ideal. The second one is an extension of a theorem of Ihara on the study of maps between Jacobians of modular curves of different levels. Here we would like to mention how helpful was for us the paper of Darmon, Diamond and Taylor [6] reviewing the work of Wiles.

Let us just mention that we tried to apply the same ideas for $p > 3$. Here in order to bring things to the previous setting we use the fact that the base-change property for automorphic representations of $GL(2)$ has been proved for cyclic extensions [23]. Using this, we can work the congruences over the totaly real field $F := \mathbb{Q}(\mu_p)^+$. However we face two problems. First the fact that we work with a prime that ramifies in $F$ puts restrictions on the freeness results that we need. Second we need to relate our defined automorhic periods over $F$ with the ones over $\mathbb{Q}$, and even stronger we need the relation to be up to $p$-adic units a problem much of the same nature that we face in our work [3]. We do not have an answer to these questions yet.

**Acknowledgements:** The author would like to thank Professor John Coates for suggesting to work on Kato's congruences and for recommending to consider the use of the Rankin-Selberg method and its $p$-adic version.

## 2 Basic Notations

Let $\mathbb{H}$ be the complex upper half plane. If we denote by $GL_2^+(\mathbb{R})$ the two by two real matrices with positive determinant, then we consider the action of them on $\mathbb{H}$ by liner fractional transformations, $z \mapsto \alpha(z) = \frac{az+b}{cz+d}$, for $\alpha = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in GL_2^+(\mathbb{R})$. We let $k \geq 1$ be an integer and we define an action of $GL_2^+(\mathbb{R})$ on functions $f : \mathbb{H} \to \mathbb{C}$ by

$$f \mapsto (f|_k[\alpha])(z) = det(\alpha)^{k/2}(cz+d)^{-k} f(\alpha(z))$$

for $\alpha = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in GL_2^+(\mathbb{R})$. We denote by $SL_2(\mathbb{Z})$ the two by two matrices with determinant 1 and integral entries. For a positive integer $N$ we have the standard notations for the subgroups of $SL_2(\mathbb{Z})$,

$$\Gamma(N) = \{\gamma \in SL_2(\mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N\}$$

$$\Gamma_0(N) = \{\gamma \in SL_2(\mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod N\}$$

$$\Gamma_1(N) = \{\gamma \in \Gamma_0(N) \mid \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod N\}$$

We write $M_k(\Gamma_1(N))$ (resp. $S_k(\Gamma_1(N))$) for the space of modular forms (resp. cusp forms) of weight $k$ with respect to $\Gamma_1(N)$. We write $M_k(\Gamma_0(N), \chi)$ (resp $S_k(\Gamma_0(N), \chi)$ for modular forms (resp. cusp forms) with respect to $\Gamma_0(N)$ and Nebentype $\chi$.

Let us consider a cusp form $f \in S_k(\Gamma_0(N), \chi)$ and a modular form $g \in M_l(\Gamma_0(N), \psi)$, for some integers $k$ and $l$ where we moreover assume $k > l$. Let us write their Fourier expansions at $\infty$ cusp as $f(z) = \sum_{n=1}^{\infty} a(n,f)q^n$ and $g(z) = \sum_{n=0}^{\infty} a(n,g)q^n$ with $q = e^{2\pi \imath z}$. We also define $f^{\rho}(z) = \sum_{n=1}^{\infty} \overline{a(n,f)}q^n \in S_k(\Gamma_0(N), \bar{\chi})$. We consider the quantities $L(f,g,s) := \sum_{n=1}^{\infty} a(n,f)a(n,g)n^{-s}$ and their Rankin-Selberg convolution, $D(f,g,s) := L_N(\chi\psi, 2s+2-k-l)L(f,g,s)$ where we have removed the Euler factors at $N$ from $L(\chi\psi, s)$. If we assume that $f$ and $g$ are actually normalized eigenforms and if we write their $L$ functions $L(f,s) = \prod_q \{(1-\alpha(q,f)q^{-s})(1-\beta(q,f)q^{-s})\}^{-1}$ and $L(g,s) = \prod_q \{(1-\alpha(q,g)q^{-s})(1-\beta(q,g)q^{-s})\}^{-1}$ then we have that

$$D(f,g,s) = \prod_q \{(1-\alpha(q,f)\alpha(q,g)q^{-s})(1-\alpha(q,f)\beta(q,g)q^{-s})\times$$
$$(1-\beta(q,f)\alpha(q,g)q^{-s})(1-\beta(q,f)\beta(q,g)q^{-s})\}^{-1}$$

# 3 $p$-adic modular forms and measures

In this section we introduce the needed background in order to obtain the $p$-adic version of the Rankin-Selberg convolution. For all this background we follow Hida's papers [13, 14]. We let $p$ be a prime number and we fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C}_p$, where $\mathbb{C}_p$ is the $p$-adic completion of $\overline{\mathbb{Q}}_p$ under the normalized $p$-adic absolute value $|\cdot|_p$ with $|p|_p = p^{-1}$. For any subring $R \subseteq \overline{\mathbb{Q}}$ we consider the $R$-modules,

$$M_k(\Gamma_0(N), \psi; R) := \{f \in M_k(\Gamma_0(N), \psi) \mid \quad f(z) = \sum_{n \geq 0} a(n,f)q^n, \ \ a(n,f) \in R\}$$

$$M_k(\Gamma_1(N); R) := \{f \in M_k(\Gamma_1(N)) \mid \quad f(z) = \sum_{n \geq 0} a(n,f)q^n, \ \ a(n,f) \in R\}$$

Moreover we define $S_k(\Gamma_0(N), \psi; R) = S_k(\Gamma_0(N), \psi) \cap M_k(\Gamma_0(N), \psi; R)$ and similar for $S_k(\Gamma_1(N); R)$. For a modular form $f \in M_k(\Gamma_1(N); \overline{\mathbb{Q}})$ it is known that one can define the $p$-adic norm of $f$, $|f|_p := sup_{n \geq 0}|a(n,f)|_p$. Let now $K_0$ be any finite extension of $\mathbb{Q}$ and write $K$ for the closure of $K_0$ in $\mathbb{C}_p$. We define the space $M_k(\Gamma_0(N), \psi; K)$ (resp. $M_k(\Gamma_1(N); K)$) to be the $p$-adic completion of the space $M_k(\Gamma_0(N), \psi; K_0)$ (resp. $M_k(\Gamma_1(N); K_0)$ with respect to the norm $|\cdot|_p$ inside $K[[q]]$ where we consider $q$ as indeterminant. Then it is known by the work of Deligne and Rapoport [8] that, $M_k(\Gamma_0(N), \psi; K) = M_k(\Gamma_0(N), \psi; K_0) \otimes_{K_0} K$, $M_k(\Gamma_1(N); K) = M_k(\Gamma_1(N); K_0) \otimes_{K_0} K$. Moreover it is known that the definition of $M_k(\Gamma_1(N); K)$ and $M_k(\Gamma_0(N), \psi; K)$ is independent of the choice of the dense subfield $K_0$. Let us now write $\mathcal{O}_K$ for the $p$-adic ring of integers of $K$. Then we define the $p$-adic integral modular forms as,

$$M_k(\Gamma_0(N), \psi; \mathcal{O}_K) := \{f \in M_k(\Gamma_0(N), \psi; K) \mid |f|_p \leq 1\} = M_k(\Gamma_0(N), \psi; K) \cap \mathcal{O}_K[[q]],$$

$$M_k(\Gamma_1(N); \mathcal{O}_K) := \{f \in M_k(\Gamma_1(N); K) \mid |f|_p \leq 1\} = M_k(\Gamma_1(N); K) \cap \mathcal{O}_K[[q]]$$

**Definition 1** ($p$-adic modular forms). *Let $A$ be either $K$ or $\mathcal{O}_K$. We consider the spaces,*

$$M_k(N;A) := \cup_{n=0}^{\infty} M_k(\Gamma_1(Np^n);A) \ \ and \ \ M_k(N,\psi;A) := \cup_{n=0}^{\infty} M_k(\Gamma_0(Np^n),\psi;A)$$

*Then we define the space of $p$-adic modular forms of $\Gamma_1(N)$, resp. of $\Gamma_0(N)$ and character $\psi$, as the completion of the above spaces with respect to the norm $|\cdot|_p$. We denote them by $\overline{M}_k(N;A)$, resp. $\overline{M}_k(N,\psi;A)$.*

We note that all the above discussion can be done considering cusp forms instead of modular forms. In particular we can consider also $p$-adic cusp forms which we will denote by $\overline{S}_k(N,A)$ and $\overline{S}_k(N,\psi;A)$.

**Remark 1** *For our later use, we mention that the space $\overline{M}_k(N,A)$ is actually independent of $k$ for $k \geq 2$, so we may also write just $\overline{M}(N;A)$, see [14].*

Now we are going to define $p$-adic Hecke operator that extend the usual ones when restricted to the space of classical modular forms. For any integer $n$ prime to $N$ we consider a matrix $\sigma_n \in \Gamma_0(N)$, such that $\sigma_n \equiv \begin{pmatrix} n^{-1} & o \\ 0 & n \end{pmatrix} \mod N$. It follows by the work of Deligne and Rapoport [8] that the action $f \mapsto f|_k\sigma_n$ on $M_k(\Gamma_1(N);K)$ is integral, that is it preserves the integral space $M_k(\Gamma_1(N);\mathcal{O}_K)$. We "define" the Hecke operators $T(\ell)$ and $S(\ell)$, for every prime $\ell$, acting on $M_k(\Gamma_1(N);K)$ by describing their action on the $q$-expansion,

$$a(n,T(\ell)f) = \begin{cases} a(\ell n,f) + \ell^{k-1}a(\frac{n}{\ell}, f|_k\sigma_\ell), & \text{if } \ell \text{ is prime to } N; \\ a(\ell n,f), & \text{otherwise.} \end{cases}$$

$$a(n,S(\ell)(f)) = \begin{cases} \ell^{k-2}a(n,f|_k\sigma_\ell), & \text{if } \ell \text{ is prime to } N; \\ 0, & \text{otherwise.} \end{cases}$$

Note that these definitions are consistent with the ones on the classical elliptic modular forms. We define the Hecke algebra $H_k(\Gamma_0(N),\psi;A)$, resp. $H_k(\Gamma_1(N);A)$), for $A$ either $K$ or $\mathcal{O}_K$ as the $A$-subalgebra of $End_A(M_k(\Gamma_0(N),\psi;A))$, resp. $End_A(M_k(\Gamma_1(N);A))$, generated by $T(\ell)$ and $S(\ell)$ for all primes $\ell$. Similarly we define $h_k(\Gamma_0(N);\psi;A)$ and $h_k(\Gamma_1(N);A)$ when we restrict the action to the space of cusp forms. Actually one has that $H_k(\Gamma_0(N),\psi;A) = H_k(\Gamma_0(N),\psi;\mathbb{Z}) \otimes_{\mathbb{Z}} A$ and similarly for the other spaces. Finally we note that when $p|N$ the action of the Hecke operators is $p$-adically integral i.e. $|Tf|_p \leq |f|_p$ for every $T \in H_k(\Gamma_1(N);\mathcal{O}_K)$.

We now define $p$-adic Hecke algebras. Notice that we have the $\mathcal{O}_K$-surjective homomorphisms induced by restriction of the Hecke operators,

$$H_k(\Gamma_0(Np^m),\psi;\mathcal{O}_K) \rightarrow H_k(\Gamma_0(Np^n),\psi;\mathcal{O}_K) \quad for \ \ m \geq n \geq 1$$

$$H_k(\Gamma_1(Np^m);\mathcal{O}_K) \rightarrow H_k(\Gamma_1(Np^n);\mathcal{O}_K) \quad for \ \ m \geq n \geq 1$$

**Definition 2** *We define the space of $p$-adic Hecke algebras $H_k(N,\psi;\mathcal{O}_K)$ (resp. $H_k(N;\mathcal{O}_K)$) by the projective limit, $\varprojlim_n H_k(\Gamma_0(N),\psi;\mathcal{O}_K)$ (resp. $\varprojlim_n H_k(\Gamma_1(N);\mathcal{O}_K)$). Similarly we define the spaces $h_k(N,\psi;\mathcal{O}_K)$ and $h_k(N;\mathcal{O}_K)$.*

By definition this operators act on the spaces $M_k(N;A)$ and $M_k(N,\psi;A)$ for $A$ equal to $K$ or $\mathcal{O}_K$. However the fact they are $p$-adically integral allow us to extend their action to the space of $p$-adic modular forms $\overline{M}_k(N;A)$ and $\overline{M}_k(N,\psi;A)$. Our next step is to define Hida's ordinary idempotent $e$ attached to the Hecke operator $T(p)$. We start with a general lemma,

**Lemma 1** *For any commutative $\mathcal{O}_K$-algebra $R$ of finite rank over $\mathcal{O}_K$ and for any $x \in R$ the limit $\lim_{n\to\infty} x^{n!}$ exists and gives an idempotent of $R$.*

**Proof** See [16] (p.201) ■.

**Definition 3** *We define an idempotent $e_n$ in $H_k(\Gamma_0(Np^n,\psi;\mathcal{O}_K)$ and in $H_k(\Gamma_1(Np^n;\mathcal{O}_K)$ by the limit $e_n = \lim_{m\to\infty} T(p)^{m!}$. Moreover we define an idempotent in $H_k(N;\mathcal{O}_K)$ and in $H_k(N,\psi;\mathcal{O}_K)$ by taking the projective limit $e = \varprojlim_n e_n$.*

We will be interested in the space $e\overline{M}_k(N,\psi;\mathcal{O}_K)$, usually called the ordinary part of $\overline{M}_k(N,\psi;\mathcal{O}_K)$ and denoted by $\overline{M}_k^{\circ}(N,\psi;\mathcal{O}_K)$. Actually this space is not that large as the following lemma indicates,

**Lemma 2** (*Hida*) *Let $C(\psi)$ be the conductor of the character $\psi$. Define positive integers $N'$ and $C(\psi)'$ by writing $N = N'p^r$ and $C(\psi) = C(\psi)'p^t$ with $(N',p) = (C(\psi)',p) = 1$. Let $s := max(t,1)$. Then,*

$$e\overline{M}_k(N,\psi;\mathcal{O}_K) \subset M_k(\Gamma_0(N'p^s),\psi;\mathcal{O}_K)$$

**Proof:** See [13].

**Definition 4** *We say that a normalized eigenform $f_0 \in S_k(\Gamma_0(N_0)\psi)$ is an ($p$-) ordinary form if,*

1. *The level $N_0$ of the form $f$ is divisible by $p$.*
2. *The Fourier coefficient $a(p,f_0)$ is a $p$-adic unit.*

The following lemma is proved in [13] (p. 168),

**Lemma 3** *Let $f \in S_k(\Gamma_0(N),\psi)$ be a newform with $k \geq 2$ and $|a(p,f)|_p = 1$. Then, there is a unique ordinary form $f_0$ of weight $k$ and character $\psi$ such that $a(n,f) = a(n,f_0)$ for all $n$ not divisible by $p$. Moreover, $f_0$ is given by,*

$$f_0(z) = \begin{cases} f(z), & \text{if } p \text{ divides } N; \\ f(z) - wf(pz), & \text{otherwise.} \end{cases}$$

*where $w$ is the unique root of $X^2 - a(p,f)X + \psi(p)p^{k-1} = 0$ with $|w|_p < 1$. Moreover in the second case i.e. $(p,N) = 1$ we have that $N_0 = Np$ and that $a(p,f_0) = u$ where $u$ is the $p$-adic unit root of the above equation.*

Let us now consider a surjective $K$-linear homomorphism $\Phi : h_k(\Gamma_0(N_0), \psi; K) \to K$ that is induced by an ordinary form $f_0$ by sending $T(n) \mapsto a(n, f_0)$. Let us moreover assume that this map is split (we will show later that in the case of interest this will be true) and induces an algebra direct decomposition, $h_k(\Gamma_0(N_0), \psi; K) \cong K \times A$ for some summand $A$ and let us denote by $1_{f_0}$ the idempotent corresponding to the first summand isomorphic to $K$. We now consider the linear form $\ell_{f_0} : \overline{S}_k(N_0, \psi; K) \to K$ defined by, $\ell_{f_0}(g) := a(1, 1_{f_0} e\ g)$. Note that, by lemma 2, the linear form is well defined.

**Proposition 1** ($Hida's linear operator$) *Assume that $K_0$ contains all the Fourier coefficients of the ordinary form $f_0$. Then, the linear form $\ell_{f_0}$ has values in $K_0$ on $S_k(\Gamma_0(N_0p^n), \psi; K_0)$ for every $n \geq 0$. Furthermore, for $g \in S_k(\Gamma_0(N_0p^n), \psi; K_0)$ we have*

$$\ell_{f_0}(g) = a(p, f_0)^{-n} p^{n(k/2)} \frac{< h_n, g >_{N_0p^n}}{< h, f_0 >_{N_0}}$$

*where $h = f_0^{\rho}|_k \begin{pmatrix} 0 & -1 \\ N_0 & 0 \end{pmatrix}$, $h_n(z) = h(p^n z)$.*

**Proof** See [13] p.175.

We note that if we consider a constant $c(f_0) \in \mathcal{O}_K$ such that $c(f_0)1_{f_0} \in h_k(\Gamma_1(N_0); \mathcal{O}_K)$ then we have an integral valued linear form $c(f_0)\ell_{f_0} : \overline{S}_k(N_0, \psi; \mathcal{O}_K) \to \mathcal{O}_K$ as the Hecke operators are $p$-adically integral

**$p$-adic modular forms valued measures:** Now we are going to define $p$-adic measures associated with $p$-adic modular forms $\overline{M}(N; \mathcal{O}_K)$ for some $N$ relative prime to $p$. Note that it follows from remark 1 that we do not need to specify the weight.

We let $X$ to be a $p$-adic space that consists of some copies of $\mathbb{Z}_p$ and of a finite product of finite groups. For our applications later $X$ is going to be just $\mathbb{Z}_p^\times \cong (1 + p\mathbb{Z}_p) \times (\mathbb{Z}/p\mathbb{Z})^\times$. Let us write $C(X; \mathcal{O}_K)$ for the space of continuous functions of $X$ with values in $\mathcal{O}_K$ and $LC(X; \mathcal{O}_K)$ for the space of locally constant functions on $X$. A measure $\mu$ on $X$ with values in the space $\overline{M}(N; \mathcal{O}_K)$ is just an $\mathcal{O}_K$-linear homomorphism from $C(X; \mathcal{O}_K)$ to $\overline{M}(N; \mathcal{O}_K)$.

Let us consider the space $Z_N := \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$ and for an element $z \in Z_N$ let us write $z_p$ for the projection of $z$ to the first component. We can define an action of $Z_N$ on the space $M_k(\Gamma_1(Np^r); \mathcal{O}_K)$ by $f \mapsto f|z := z_p^k f|_k \sigma_z$ with $\sigma_z$ as defined above. This action can be extended to $\overline{M}(N, \mathcal{O}_K)$ (see [14] p. 10).

**Definition 5** *(see [14]) We say that a $p$-adic measure $\mu : C(X; O_K) \to \overline{M}(N; O_K)$ is arithmetic if the following three conditions are satisfied,*

1. *There exists positive integer $k$ such that for every $\phi \in LC(X; \mathcal{O}_K)$,*

$$\mu(\phi) \in M_k(Np^\infty; \mathcal{O}_K)$$

*We will call $k$ the weight of $\mu$.*

2. *There are continuous action $Z_N \times X \to X$ and a finite order character $\xi : Z_N \to \mathcal{O}_K^\times$ such that $\mu(\phi)|z = z_p^k \xi(z)\mu(\phi(z \cdot x))$ for every $\phi \in C(X; O_K)$, where $k$ the weight of $\mu$. We then say that the arithmetic measure is of character $\xi$.*

*We say that the measure is cuspidal if $\mu$ actually takes values in $\overline{S}(N; \mathcal{O}_K)$.*

We are interested in attaching arithmetic measures to a given modular form. Given a modular form $f \in M_k(\Gamma_0(N), \chi; \mathcal{O}_K)$ with $q$-expansion $f(z) = \sum_{n \geq 0} a(n, f)q^n$ we can associate a measure $d\mu_f$ on $X := \mathbb{Z}_p^\times$ by,

$$d\mu_f(\phi) \mapsto \sum_{n \geq 1} \phi(n)a(n, f)q^n, \quad \phi \in C(X; \mathcal{O}_K)$$

where we define the action of $Z_N$ on $\mathbb{Z}_p^\times$ by $z \cdot x \mapsto z_p^2 x$. From the following lemma due to Shimura we conclude that $d\mu_f$ is an arithmetic measure of weight $k$ and character $\chi$.

**Lemma 4** *Let $g = \sum_{n=0}^\infty b(n, g)q^n \in M_k(\Gamma_0(N), \omega)$ and $\phi$ an arbitrary function on $Y_m =$**Z**$/Np^m$**Z**. Define $g(\phi) := \sum_{n=0}^\infty \phi(n)b(n, g)q^n$. Then for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N^2p^{2m})$, we have the following transformation formula,*

$$g(\phi)|_k \gamma = \omega(d)g(\phi_a)$$

*where $\phi_a(y) = \phi(a^{-2}y)$ for all $y \in Y_m =$**Z**$/Np^m$**Z**.*

**Proof** See [13] (p. 190) ∎

By a result of Hida in [14] (p. 24 corollary 2.3) it follows that actually the measure $\mu_f$, on $\mathbb{Z}_p^\times$, is cuspidal.

**Eisenstein measure and convolution:** Of particular importance for us is the existence, which follows from [21], of the following arithmetic measure of weight one, $dE : C(Z_L; \mathcal{O}_K) \to \bar{S}(L; \mathcal{O}_K)$ defined by,

$$2 \int_{Z_L} \phi(z)dE = \sum_{\substack{n=1 \\ (n,p)=1}}^\infty \left( \sum_{\substack{d|n \\ (d,L)=1}} sgn(d)\phi(d) \right) q^n \in \mathcal{O}_K[[q]]$$

We call this the Eisenstein-Katz measure. For a general arithmetic measure $\mu_g$ of $\mathbb{Z}_p^\times$ associated to a modular form of weight $\ell$ and character $\psi$ we can define a convolution operation, see for example [13, 26], of $\mu_g$ and $dE$. We consider the action of $Z_L$ on $C(\mathbb{Z}_p^\times; \mathcal{O}_K)$ by $(z \star \phi)(x) := \psi(z)z_p^\ell \phi(z_p^2 x)$ for $z \in Z_L$ and $\phi \in C(\mathbb{Z}_p^\times; \mathcal{O}_K)$. For a given integer $k \geq \ell$ and a finite order character $\chi : Z_L \to \mathbb{C}^\times$ we define the arithmetic measure $(\mu_g * dE)_{\chi,k} : C(\mathbb{Z}_p^\times; \mathcal{O}_K) \to \overline{S}(L; \mathcal{O}_K)$ as

$$\int_{\mathbb{Z}_p^\times} \phi(x)(\mu_g^L * dE)_{\chi,k} := \int_{\mathbb{Z}_p^\times} \int_{Z_L} \chi(z)z_p^{k-1}(z^{-1} \star \phi)(x)dE(z)d\mu_g^L(x)$$

# 4 $p$-adic Rankin-Selberg convolution

Now we have collected all the needed background from the theory of $p$-adic modular forms and measures to introduce $p$-adic Rankin-Selberg convolution. In this section we state and prove a simplified version, sufficient for our purposes, of a theorem of Hida, as for example stated in [14] theorem 5.1.

Let $f \in S_k(\Gamma_0(N), \chi)$ be a normalized eigenform with $|a(p, f)|_p = 1$ and $(N, p) = 1$. Write $f_0 \in \Gamma_0(Np), \chi)$ for the corresponding ordinary form. We recall that $f_0 = f - \frac{\chi(p)p^{k-1}}{u} f|[p]$ where $f|[p](z) := f(pz)$ and $u$ the root of $X^2 - a(p, f)X + \chi(p)p^{k-1} = 0$ which is a $p$-adic unit. Let $g \in M_\ell(\Gamma_0(Jp^\alpha), \psi)$ with $(J, p) = 1$ and $k > \ell$. Consider the cuspidal arithmetic measure $\mu_g$ on $X := \mathbb{Z}_p^\times$ that we can attach to $g$ from the previous section. Now we assume that we can attach to $f_0$ a linear form $\ell_{f_0} : \overline{S}_k(N; \mathcal{O}_K) \to K$ as in the previous section. We also consider a constant $c(f)$ such that $c(f)\ell_f$ takes integral values. Then we have,

**Theorem 1** ($p$-*adic Rankin-Selberg convolution*) *With notation as above, there is a measure $\mu_{f\times g} : \mathbb{Z}_p^\times \to \mathcal{O}_K$ such that for any finite order character $\phi$ on $\mathbb{Z}_p^\times$,*

$$\int_{\mathbb{Z}_p^\times} \phi d\mu_{f\times g} = c(f)(-1)^k t a(p, f_0)^{1-\beta} p^{\beta\ell/2} p^{\frac{2-k}{2}\beta} \frac{D(f_0, \mu_g(\phi)|_\ell \tau_\beta, \ell)}{2^{k+\ell}\pi^{\ell+1}\imath^{k+\ell} < f_0^\rho |_k \tau_{Np}, f_0 >_{Np}}$$

*where,*

$$t = (-1)^k l.c.m(N, J) N^{k/2} J^{\ell/2} \Gamma(\ell)$$

*and $\beta$ is such that $\mu(\phi) \in M_\ell(\Gamma_1(Jp^\beta))$ and* $\tau_\beta = \begin{pmatrix} 0 & -1 \\ Jp^\beta & 0 \end{pmatrix}$

**Remark 2** *This is a special case of a more general result of Hida. First of all using Shimura's differential operators he can show that the above p-adic measure interpolates the rest of the critical values of the Rankin-Selberg L-function i.e. $D(\ell + m, f_0, \mu(\phi)|_\ell \tau_\beta)$ for $0 \le m \le k - \ell$. Second, and most important, Hida can construct $p$-adic measures that interpolate families of modular forms (usually called $\Lambda$-adic modular forms), in both variables of the Rankin-Selberg product (under some ordinarity assumptions also on the second variable).*

We give the proof of the above theorem following Hida as in [14] (page 76). The explicit construction of the measure $\mu_{f\times g}$ is important for our purposes.

**Proof:** Let us denote by $L$ the least common multiple of $N$ and $J$. We consider the Eisenstein-Katz measure $dE : C(Z_L; \mathcal{O}_K) \to \overline{S}(L; \mathcal{O}_K)$ that we have introduced in the previous section. Recall that is defined as,

$$2\int_{Z_L} \phi(z) dE = \sum_{\substack{n=1 \\ (n,p)=1}}^{\infty} \left( \sum_{\substack{d|n \\ (d,L)=1}} sgn(d)\phi(d) \right) q^n \in \mathcal{O}_K[[q]]$$

and $Z_L = \mathbb{Z}_p^\times \times (\mathbb{Z}/L\mathbb{Z})^\times$. We also modify the arithmetic measure $\mu_g$ by defining a new one, $\mu_g^L(\phi) := \mu_g(\phi) \mid [L/J]$ where $[L/J] : \overline{S}(J;O_K) \to \overline{S}(L;O_K)$, as $[L/J](\sum_{n\geq 1} a(n,g)q^n) \mapsto \sum_{n\geq 1} a(n,g)q^{n\frac{L}{J}}$ , and hence $\mu_g^L$ is again an arithmetic measure of weight $\ell$ and character $\psi$. Recall that we have defined an action of $Z_L$ on $\mathbb{Z}_p^\times$ as $z\cdot x \mapsto z_p^2 x$ and by the previous section we can consider the convoluted measure $(\mu_g^L * dE)_{\chi,k}$, which we recall is defined by,

$$\int_{\mathbb{Z}_p^\times} \phi(x)(\mu_g^L * dE)_{\chi,k} := \int_{\mathbb{Z}_p^\times}\int_{Z_L} \chi(z)z_p^{k-1}(z^{-1}\star\phi)(x)dE(z)d\mu_g^L(x)$$

Now we define the measure $\mu_{f\times g}$ as,

$$\int_{\mathbb{Z}_p^\times} \phi d\mu_{f\times g} := c(f)\circ \ell_{f_0}\circ Tr_{L/N}\circ e(\int_{\mathbb{Z}_p^\times}\phi(\mu_g^L * dE)_{\chi,k})$$

Here $Tr_{L/N} : \overline{M}(L;\mathcal{O}_K) \to \overline{N}(N;\mathcal{O}_K)$ is the trace operator, see [26]. We do not need to give its detailed definition but just mention that when restricted to the classical modular forms satisfy the usual property; for $f \in S_k(\Gamma_1(N);\mathcal{O}_K)$ and $g \in M_k(\Gamma_1(L));\mathcal{O}_K)$ we have that, $< f, Tr_{L/N}g >_N =< (L/N)^k f|[L/N], g >_L$. Let now $\phi$ be a finite order character on $\mathbb{Z}_p^\times$. We compute the value of our measure on $\phi$. We have,

$$\int_{\mathbb{Z}_p^\times} \phi(\mu_g^L * dE)_{\chi,k} = \int_{\mathbb{Z}_p^\times}\int_{Z_L} \chi(z)z_p^{k-1}(z^{-1}\star\phi)(x)dE(z)d\mu_g^L(x) =$$

$$= \int_{\mathbb{Z}_p^\times}\int_{Z_L} \chi(z)z_p^{k-1}\psi(z)^{-1}z_p^{-\ell}\phi(z_p)^{-2}\phi(x)dE(z)d\mu_g^L(x) =$$

$$= \int_{\mathbb{Z}_p^\times}\int_{Z_L} \chi(z)z_p^{k-\ell-1}\psi(z)^{-1}\phi(z_p)^{-2}\phi(x)dE(z)d\mu_g^L =$$

$$= \left(\int_{\mathbb{Z}_p^\times}\phi(x)d\mu_g^L\right)\cdot\left(\int_{Z_L}\chi\psi^{-1}(z)\phi^{-2}(z_p)z_p^{k-\ell-1}dE(z)\right)$$

Evaluating the Eisenstein measure we get,

$$\int_{Z_L}\chi\psi^{-1}(z)\phi^{-2}(z_p)z_p^{k-\ell-1}dE = E_{k-\ell,Lp}(\chi\psi^{-1}\phi_p) \mid \imath_p$$

where $\phi_p(z) = \phi^{-2}(z_p)$ and,

$$E_{m,M}(\theta) := \frac{1}{2}L_M(1-m,\theta) + \sum_{n=1}^{\infty}\left(\sum_{\substack{0<d|n\\(d,Mp)=1}}\theta(d)d^{m-1}\right)q^n$$

an Eisenstein series in $M_m(\Gamma_0(M),\theta)$. We consider now the projection to the ordinary part. By the property $e(f\cdot g|\imath_p) = e(f|\imath_p\cdot g)$ (see [14], p.24) and since $\mu_g^L(\phi)|\imath_p =$

$\mu_g^L(\phi)$ as it is measure over $\mathbb{Z}_p^\times$ we have that,

$$e(\int_{\mathbb{Z}_p^\times}\phi(\mu_g^L * dE)_{\chi,k}) = \left(\int_{\mathbb{Z}_p^\times}\phi(x)d\mu_g^L\right)\cdot E_{k-\ell,Lp}(\chi\psi^{-1}\phi_p)$$

Applying the explicit formula for the linear form $\ell_{f_0}$ and after writing $h := (\int_{\mathbb{Z}_p^\times}\phi(\mu_g^L * dE)_{\chi,k} \in S_k(\Gamma_0(Np^\beta)),\chi)$ we have,

$$\int_{\mathbb{Z}_p^\times}\phi d\mu_{f\times g} = c(f)a(p,f_0)^{1-\beta}p^{(\beta-1)(k/2)}\frac{< (f_0^\rho\mid_k \tau_{Np}) \mid [p^{\beta-1}], Tr_{L/N}(h) >_{Np^\beta}}{< f_0^\rho \mid_k \tau_{Np}, f_0 >_{Np}}$$

We claim the equality,

$$< (f_0^\rho\mid_k \tau_{Np}) \mid [p^{\beta-1}], Tr_{L/N}(h) >_{Np^\beta} = (L/N)^{k/2}p^{k(1-\beta)/2} < f_0^\rho \mid \tau_{Lp^\beta}, h >_{Lp^\beta}$$

Indeed by the property of the trace operator that we described above we have,

$$\begin{aligned}< (f_0^\rho\mid_k \tau_{Np}) \mid [p^{\beta-1}], Tr_{L/N}(h) >_{Np^\beta} = (L/N)^k < (f_0^\rho\mid_k \tau_{Np}) \mid \\ [Lp^{\beta-1}/N], h >_{Lp^\beta} \\ = (L/N)^{k/2}p^{k(1-\beta)/2} < f_0^\rho \mid \tau_{Lp^\beta}, h >_{Lp^\beta}\end{aligned}$$

Hence the evaluation of the measure now reads as,

$$\int_{\mathbb{Z}_p^\times}\phi d\mu_{f\times g} = c(f)a(p,f_0)^{1-\beta}p^{(\beta-1)(k/2)}\frac{(L/N)^{k/2}p^{k(1-\beta)/2} < f_0^\rho \mid \tau_{Lp^\beta}, h >_{Lp^\beta}}{< f_0^\rho \mid_k \tau_{Np}, f_0 >_{Np}}$$

We note that we can write ,

$$\mu_g^L(\phi) = (-1)^\ell (L/J)^{-\ell/2}(\mu_g(\phi)|_\ell \tau_{Jp^\beta})|_\ell \tau_{Lp^\beta}$$

The following proposition, which is taken from [14] p. 63 allow us to conclude the proof.

**Proposition 2** *Let* $h_1 \in S_k(\Gamma_0(Lp^\beta),\psi)$ *and* $h_2 \in M_\ell(\Gamma_0(Lp^\beta),\xi)$. *Then,*

$$D(h_1,h_2,\ell) = t' < h_1^\rho|_k\tau_{Lp^\beta}, (h_2|_\ell\tau_{Lp^\beta})(E_{k-\ell,Lp}(\xi\psi)) >_{Lp^\beta}$$

*where* $t' := 2^{k+\ell}\pi^{\ell+1}(Lp^\beta)^{\frac{1}{2}(k-\ell-2)}(\sqrt{-1})^{\ell-k}(\Gamma(\ell))^{-1}$

# 5 Towards the congruences

In this section we obtain a first form of the congruences claimed in the introduction.

## 5.1 The case $p = 3$

We start with some generalities. Let $K/\mathbb{Q}$ be a quadratic imaginary extension of discriminant $D$ and non-trivial character $\epsilon_D$. Let $\chi^* : \mathbb{A}_K^\times/K^\times \to \mathbb{C}^\times$ be a finite order Hecke character corresponding by class field theory to a Galois character $\chi : Gal(K(\mathrm{f}_\chi)/K) \to \mathbb{C}^\times$ where $\mathrm{f}_\chi$ the conductor of $\chi$ and $K(\mathrm{f}_\chi)$ the ray class field for the ideal $\mathrm{f}_\chi$. We will also write $\chi$ for the ideal character corresponding to $\chi^*$. Consider the series $g_\chi(z) = \sum_{\mathrm{a}\subset\mathcal{O}_K}\chi(\mathrm{a})q^{N(\mathrm{a})}$ if $\chi$ is not the trivial character where $q = e^{2\pi\imath z}$ and $\chi(\mathrm{a}) = 0$ if $(\mathrm{a}, \mathrm{f}_\chi) \neq 1$. In case $\chi$ is the trivial character we define $g_1(z) = \frac{1}{2}L(0,\epsilon_D) + \sum_{\mathrm{a}\subset\mathcal{O}_K} q^{N(\mathrm{a})}$. By automorphic induction we have that $g_\chi(z) \in M_1(\Gamma_0(|D|N(\mathrm{f}), \epsilon_D\chi|_{\mathbb{Z}})$ where by $\chi|_{\mathbb{Z}}$ we mean the character obtained by restricting $\chi$ to ideals in $\mathbb{Z}$. Moreover it is known that for $\chi$ non-trivial we have that $g_\chi(z) \in S_1(\Gamma_0(|D|N(\mathrm{f}), \epsilon_D\chi|_{\mathbb{Z}})$ is a primitive form.

Let us write $p$ for the prime number 3. We consider the field $\mathbb{Q}(\mu_p)/\mathbb{Q}$ and we write $\mathfrak{p}$ for the unique prime above $p$ in it. Let us now denote by $\chi$ any of the two non-trivial character of the cyclic cubic extension $\mathbb{Q}(\mu_p, \sqrt[p]{m})/\mathbb{Q}(\mu_p)$ for $m$ a power free integer and $(m,p) = 1$. Note that $\chi \equiv 1 \mod \mathfrak{p}$. We consider the induced representation $\rho := Ind_{\mathbb{Q}}^K(\chi)$, a two dimensional Artin representation $\rho : S_3 \cong Gal(\mathbb{Q}(\mu_p, \sqrt[p]{m})/\mathbb{Q}) \to GL_2(\mathbb{Z})$. We write $g_\rho$ for the corresponding newform obtained from the discussion above with $g_\rho \in S_1(\Gamma_0(m^2p^r), \epsilon_p\chi|_{\mathbb{Z}})$ where $r = 1$ if $\chi$ does not ramify at $p$ and $r = 3$ if it does. Note that actually $\chi|_{\mathbb{Z}}$ is the trivial character. Finally let us also write $g_\sigma$ for the Eisenstein series $g_1$. Then, $g_\sigma \in M_1(\Gamma_0(p), \epsilon_p)$.

We associate $p$-adic arithmetic measures to our modular forms $g_\sigma$ and $g_\rho$. We modify $g_\sigma$ and consider the modular form $g_{\sigma_{(m)}} := g_\sigma \mid \imath_m \in M_1(\Gamma_0(m^2p, \epsilon_p)$, with $\imath_m$ the trivial character modulo $m$, i.e we remove the "Euler factors" at the primes dividing $m$. We now consider the associated arithmetic measures on $\mathbb{Z}_p^\times$. For $\phi \in C(\mathbb{Z}_p^\times; \mathbb{Z}_p)$ we have,

$$d\mu_\rho : \phi \mapsto \sum_{n=1}^{\infty} \phi(n)a(n, g_\rho)q^n$$

$$d\mu_{\sigma_{(m)}} : \phi \mapsto \sum_{n=1}^{\infty} \phi(n)a(n, g_{\sigma_{(m)}})q^n$$

Note that by construction we have that for any $\phi \in C(\mathbb{Z}_p^\times, \mathbb{Z}_p)$,

$$\int_{\mathbb{Z}_p^\times} \phi d\mu_\rho \equiv \int_{\mathbb{Z}_p^\times} \phi d\mu_{\sigma_{(m)}} \mod p$$

where the meaning of the congruences here is term by term i.e. $\phi(n)a(n, g_\rho) \equiv \phi(n)a(n, g_{\sigma_m}) \mod p$ for all $n$.

Let now $E/\mathbb{Q}$ be an elliptic curve over $\mathbb{Q}$ with conductor $N$ and with good ordinary reduction at $p$. Recall that we are assuming that $E[p]$ is an irreducible $G_{\mathbb{Q}}$-module and moreover the Artin conductor of the representation $\overline{\rho}_{E,p} : G_{\mathbb{Q}} \to Aut(E[p])$ is equal to $N = N_E$. Let us write $f \in S_2(\Gamma_0(N); \mathbb{Q})$ for the primitive form associated to $E$. The assumption of the good ordinary reduction at $p$ implies that $|a(p, f)|_p = 1$ where $f(z) = \sum_{n\geq 1} a(n, f)q^n$ and of course that $(N, p) = 1$. Let us now write

$f_0 \in S_2(\Gamma_0(Np);\mathbb{Q})$ for the ordinary form that we can associate to $f$ by lemma 3 and $\tilde{f}_0 \in S_2(\Gamma_0(Npm^2);\mathbb{Q})$ for the normalized eigenform that we obtain after removing the Euler factors at $q|m$, that is $\tilde{f}_0 = f_0|\imath_m$. We now consider the map $h_2(\Gamma_0(Nm^2p);\mathbb{Z}_p) \rightarrow \mathbb{Z}_p$ induced by $T(n) \mapsto a(n,\tilde{f}_0)$. Later we will prove that actually this map, under our assumptions, induces a decomposition,

$$h_2(\Gamma_0(Nm^2p);\mathbb{Q}_p) = \mathbb{Q}_p \oplus A$$

Let us write $1_{\tilde{f}_0}$ for the idempotent attached to the first summand. Moreover we consider a constant $c(f,m) \in \mathbb{Z}_p$, defined up to $p$-adic units, such that $c(f,m)1_{\tilde{f}_0} \in h_2(\Gamma_0(Nm^2p;\mathbb{Z}_p)$. Let us now write $L$ for $Nm^2$. We denote by $dE_{2,id} = dE : C(Z_L;\mathbb{Z}_p) \rightarrow \overline{S}(L;\mathbb{Z}_p)$ the Eisenstein-Katz measure on $Z_L$. Recall also that for any arithmetic measure $d\mu : C(\mathbb{Z}_p^\times;\mathbb{Z}_p) \rightarrow \overline{S}(M;\mathbb{Z}_p)$ with $M \mid L$ we have defined another arithmetic measure $d\mu^L$ with values in $\overline{S}(L,\mathbb{Z}_p)$ by applying the operator $[L/M]$.

**Lemma 5** *Let $\phi \in C(\mathbb{Z}_p^\times;\mathcal{O}_K^\times)$ be a character of finite order. Consider the measures $d\mu^L_{g_\chi} * dE$ and $d\mu^L_{\sigma_{(m)}} * dE$. Then we have,*

$$\left| \int_{\mathbb{Z}_p^\times} \phi\ (d\mu^L_{g_\rho} * dE) - \int_{\mathbb{Z}_p^\times} \phi\ (d\mu^L_{\sigma_{(m)}} * dE) \right|_p < 1$$

**Proof** By the calculations we did in the previous section for the proof of Hida's $p$-adic Rankin-Selberg theorem we have,

$$\int_{\mathbb{Z}_p^\times} \phi\ (d\mu^L_{g_\sigma} * dE) = (\int_{\mathbb{Z}_p^\times} \phi d\mu^L_{g_\rho})(\int_{Z_L} \epsilon_p(z)\phi_p(z)dE)$$

and similarly,

$$\int_{\mathbb{Z}_p^\times} \phi\ (d\mu^L_{g_{\sigma_{(m)}}} * dE) = (\int_{\mathbb{Z}_p^\times} \phi d\mu^L_{g_{\sigma_{(m)}}})(\int_{Z_L} \epsilon_p(z)\phi_p(z)dE)$$

with $\phi_p(z) = \phi^{-2}(z_p)$. The lemma now follows from the facts that $dE$ is an integral measure and $|\mu^L_{g_\sigma}(\phi) - \mu^L_{g_{\sigma_{(m)}}}(\phi)|_p < 1$ as the operator $[N]$ preserves congruences. ■

Now we are ready to prove a first type of congruences. Let us write $u$ for $a(p,f_0)$ and define $w$ by $uw = p$. We have,

**Theorem 2** *Consider the quantities,*

$$R(\rho) := c(f,m)\alpha(\rho)\frac{P_p(\rho,u^{-1})}{P_p(\rho,w^{-1})}\frac{D_{\{p,q|m\}}(f,g_\rho,1)}{\pi^2 i < \tilde{f}_0|\tau_{Lp},\tilde{f}_0 >_{Lp}}$$

*and*

$$R(\sigma) := c(f,m)\alpha(\sigma)\frac{P_p(\sigma,u^{-1})}{P_p(\sigma,w^{-1})}\frac{D_{\{p,q|m\}}(f,g_\sigma,1)}{\pi^2 i < \tilde{f}_0|\tau_{Lp},\tilde{f}_0 >_{Lp}}$$

*where, $\alpha(\rho) := e_p(\rho)u^{-v_p(N_\rho)}$, $\alpha(\sigma) := e_p(\sigma)u^{-v_p(N_\sigma)}$ with $\sigma := 1 \oplus \epsilon_p$ the Artin representation induced by the trivial character, $e_p(\cdot)$ local epsilon factor and $v_p(N_\rho)$*

*the p-adic valuation of the conductor of the Artin representation. Then with the assumptions as above $R(\rho)$ and $R(\sigma)$ are $p$-adic integers and,*

$$R(\rho) \equiv R(\sigma) \mod p.$$

Here we would like to remind the reader that under the assumption of the elliptic curve being semi-stable we have that $D(s, f, g_\rho) = L(E_f, \rho, s)$.

**Proof** We claim that for any character $\phi \in C(\mathbb{Z}_p^\times; \mathcal{O}^\times)$ we have that,

$$|\int_{\mathbb{Z}_p^\times} \phi d\mu_{\tilde{f}_0, g_\rho} - \int_{\mathbb{Z}_p^\times} \phi d\mu_{\tilde{f}_0, g_{\sigma_{(m)}}}|_p < 1$$

By definition we have,

$$\int_{\mathbb{Z}_p^\times} \phi d\mu_{\tilde{f}_0, g_\rho} = c(f,m)\ell_{\tilde{f}_0} \circ e(\int_{\mathbb{Z}_p^\times} \phi \ (d\mu^L_{g_\rho} * dE))$$

$$\int_{\mathbb{Z}_p^\times} \phi d\mu_{\tilde{f}_0, g_\sigma} = c(f,m)\ell_{\tilde{f}_0} \circ e(\int_{\mathbb{Z}_p^\times} \phi \ (d\mu^L_{g_\sigma} * dE))$$

Note that the trace operator is now just the identity. By the definition of the linear form $\ell_{\tilde{f}_0}$ we have that $c(f,m)\ell_{\tilde{f}_0}(e\,h) = a(1, c(f,m)1_{\tilde{f}_0} e\,h)$ for $h \in \overline{S}(L; \mathbb{Z}_p)$. But we have that $|e\,h|_p \leq |h|_p$ and also $|c(f,m)1_{\tilde{f}_0}\, e\,h|_p \leq |e\,h|_p$ and hence by lemma 5 we establish the claim. Now in order to obtain the congruences we evaluate both measures at the trivial character $\imath_p$ modulo $p$ and hence we have,

$$\int_{\mathbb{Z}_p^\times} \imath_p d\mu_{\tilde{f}_0, g_\rho} \equiv \int_{\mathbb{Z}_p^\times} \imath_p d\mu_{\tilde{f}_0, g_{\sigma_{(m)}}} \mod p$$

We now work both sides of the above equation. We start with the left hand side. By theorem 1 we have,

$$\int_{\mathbb{Z}_p^\times} \imath_p d\mu_{\tilde{f}_0, g_\rho} = c(f,m) u t p^{\beta/2} u^{-\beta} \frac{D(\tilde{f}_0, \mu_{g_\rho}(\imath_p, 1)|_1 \tau_\beta)}{\pi^2 i < \tilde{f}_0|_2 \tau_{Lp}, \tilde{f}_0 >_{Lp}}$$

where $\beta$ is such that $\mu_{g_\rho}(\imath_p) = g_\chi|\imath_p \in S_1(\Gamma_1(m^2 p^\beta)$. We consider the Rankin-Selberg product $D(\tilde{f}_0, \mu_{g_\rho}(\imath_p)|_1 \tau_\beta, 1) = D(\tilde{f}_0, g_\rho|\imath_p)|_1 \tau_\beta, 1)$. We will write $g$ for $g_\rho$ and $M$ for $m^2$. Let us assume first that the character $\chi$ is not ramified above $p$ and hence $\beta = 2$. We can write in this case $g|\imath_p = g - a(p,g)g|[p] = g - g|[p]$ as $a(p,g) = 1$. We apply $\tau_{Mp^2} = \begin{pmatrix} 0 & -1 \\ Mp^2 & 0 \end{pmatrix}$ to the above equation and we use the fact that $g \in \Gamma_1(Mp)$ is a primitive form of level $Mp$ and hence satisfies $g|_1 \begin{pmatrix} 0 & -1 \\ Mp & 0 \end{pmatrix} = W(g) g^\rho = W(g) g$, as $g$ has rational coefficients. The quantity $W(g)$ is usually called the root number of $g$. We have,

$$(g|\imath_p)|_1 \tau_{Mp^2} = g|_1 \tau_{Mp^2} - g|[p]|_1 \tau_{Mp^2} =$$

$$= g|_1 \begin{pmatrix} 0 & -1 \\ Mp & 0 \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} - p^{-\frac{1}{2}} g|_1 \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ Mp^2 & 0 \end{pmatrix} =$$

$$= W(g) g|_1 \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} - p^{-\frac{1}{2}} g|_1 \begin{pmatrix} 0 & -1 \\ Mp & 0 \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} =$$

$$= p^{\frac{1}{2}} W(g) g|[p] - W(g) p^{-\frac{1}{2}} g = -p^{-\frac{1}{2}} W(g)(g - pg|[p])$$

Hence we get that,

$$D(f_0, (g|\imath_p)|_1 \tau_{Mp^2}, 1) = -p^{-\frac{1}{2}} W(g)(1 - a(p, f_0)) D(f_0, g, 1) = p^{-\frac{1}{2}} W(g) u P_p(g, u^{-1}) D(f_0, g, 1)$$

Moreover we note that $D(\tilde{f}_0, g, 1) = P_p(g, w^{-1})^{-1} D_{\{p,q|m\}}(f, g, 1)$ where we have removed the Euler factor at $p$ and $q|m$ from the primitive Rankin-Selberg product $D(f, g, 1)$. Also, Balister in [1] p.17 has computed the local epsilon factors of $\rho$ from where we get $e_p(\rho) = p^{\frac{1}{2}} W_p(g)$ and $W_q(g) = 1$ for $q|m$ and hence we conclude,

$$\int_{\mathbb{Z}_p^\times} \imath_p d\mu_{\tilde{f}_0, g_\rho} = c(f, m) u t e_p(\rho) u^{-1} \frac{P_p(\rho, u^{-1})}{P_p(\rho, w^{-1})} \frac{D_{\{p,q|m\}}(f, g, 1)}{\pi^2 i < \tilde{f}_0|_2 \tau_{Lp}, \tilde{f}_0 >_{Lp}} \in \mathbb{Z}_p$$

The case where $\chi$ is ramified at $p$ is easier as $g|\imath_p = g$ since $P_p(\rho, X) = 1$ and hence we can use directly the action of $\tau_{Mp^3}$. Hence also in this case we get,

$$\int_{\mathbb{Z}_p^\times} \imath_p d\mu_{\tilde{f}_0, g_\rho} = c(f) u t e_p(\rho) u^{-3} \frac{D_{\{p,q|m\}}(f, g, 1)}{\pi^2 i < \tilde{f}_0|_2 \tau_{Lp}, \tilde{f}_0 >_{Lp}} \in \mathbb{Z}_p$$

We now work the right hand side of the congruences. We have that,

$$E_1(\epsilon_p)(z) := g_\sigma(z) = \frac{1}{2} L(0, \epsilon_p) + \sum_{n=1}^{\infty} (\sum_{0<d|n} \epsilon_p(d)) q^n \in M_1(\Gamma_0(p), \epsilon_p)$$

where we write, as always, $\epsilon_p$ for the non-trivial character of $Gal(\mathbb{Q}(\mu_3)/\mathbb{Q})$. We consider now the imprimitive Rankin-Selberg $L$-function $D(\tilde{f}_0, g_1|\imath_m|\imath_p|_1 \tau_{m^2p^2}, 1)$ where $\tau_{m^2p^2} = \begin{pmatrix} 0 & -1 \\ m^2p^2 & 0 \end{pmatrix}$. We can consider each prime separately, i.e. first we consider the quantity $D(\tilde{f}_0, g_1|\imath_p|\tau_p, 1)$ with $\tau_p = \begin{pmatrix} 0 & -1 \\ p^2 & 0 \end{pmatrix}$ and then the quantity $D(\tilde{f}_0, g_1|\imath_q|\tau_q, 1)$ with $\tau_q = \begin{pmatrix} 0 & -1 \\ pq^2 & 0 \end{pmatrix}$ for $q|m$. Let assume this and do the calculations and at the end we return to this point. As before we can write $E_1(\epsilon_p) \mid \imath_p = E_1(\epsilon_p) - E_1(\epsilon_p)|_1[p]$. Working as above we have,

$$E_1(\epsilon_p) \mid \imath_p \mid \begin{pmatrix} 0 & -1 \\ p^2 & 0 \end{pmatrix} = p^{-\frac{1}{2}} W(E_1(\epsilon_p))(E_1(\epsilon_p) - pE_1(\epsilon_p) \mid [p])$$

Hence we have,

$$D(\tilde{f}_0, E_1(\epsilon_p)|\imath_p|_1 \tau_p, 1) = p^{-\frac{1}{2}} W(E(\epsilon_p)) a(p, \tilde{f}_0)(1 - a(p, \tilde{f}_0)^{-1}) D(\tilde{f}_0, E_1(\epsilon_p), 1)$$

But for the Eisenstein series $E_1(\epsilon_p)$ we know that $D(\tilde{f}_0, E_1(\epsilon_p), 1) = L(\tilde{f}_0, 1)L(\tilde{f}_0, \epsilon_p, 1)$ or equivalently $D(\tilde{f}_0, E_1(\epsilon_p), 1) = (1 - a(p, \tilde{f}_0)p^{-1})^{-1} L_{\{p\}}(\tilde{f}, 1)L_{\{p\}}(\tilde{f}, \epsilon_p, 1)$. Recall that we have defined $u := a(p, \tilde{f}_0) = a(p, f_0)$ and $uw = p$ we finally get,

$$D(\tilde{f}_0, E_1(\epsilon_p, 1)|\imath_p|_1\tau_p) = p^{-\frac{1}{2}} W(E(\epsilon_p))u\frac{1-u^{-1}}{1-w^{-1}} L_{\{p,q|m\}}(f, 1)L_{\{p,q|m\}}(f, \epsilon_p, 1)$$

We now consider a prime $q \mid m$. Now we write just $g$ for $g_1 = E_1(\epsilon_p)$. As $q \neq p$, we have $g \mid \imath_q = g - a(q, g)g \mid [q] + \epsilon_p(q)g \mid [q^2]$. Now we apply the operator $\tau_q = \begin{pmatrix} 0 & -1 \\ q^2p & 0 \end{pmatrix}$. Doing the calculations as before we get $g|\imath_q \ |_1 \ \tau_q = q^{-1}\epsilon_p(q)W(g)(g - \epsilon_p(q)a(q, g)qg \mid [q] + \epsilon_p(q)q^2 g \mid [q^2])$. But note that since $L(s, \tilde{f}_0)$ has no Euler factors at $q \mid m$ we have that $D(s, \tilde{f}_0, g|[q^r]) = 0$ for any $r \geq 1$. So we obtain $D(\tilde{f}_0, g|\imath_q \mid \tau_q, 1) = q^{-1}\epsilon_p(q)W(g)D(\tilde{f}_0, g, 1)$. Putting all together and noticing that $q^{-1}\epsilon_p(q) \equiv 1 \mod p$ and $W_q(g) = 1$ we get,

$$\int_{\mathbb{Z}_p^\times} \imath_p d\mu_{f,g_{\sigma_m}} = c(f, m)ute_p(\sigma)u^{-1}\frac{P_p(\sigma, u^{-1})}{P_p(\sigma, w^{-1})}\frac{D_{\{p,q|m\}}(f, g_\sigma, 1)}{\pi^2 i < \tilde{f}_0|_2\tau_{Lp}, \tilde{f}_0 >_{Lp}} \in \mathbb{Z}_p$$

The fact that $|ut|_p = 1$ allows us to conclude the proof of the theorem. Let us now also justify our claim that we can work each prime separately. For simplicity we do the case of $m = q$ but we will become obvious how one obtains the general case. So with $g$ as above we have,

$$g|i_{pq} = (g - g|[p]) - a(q, g)((g - g|[p]))|[q] + \epsilon_p(q)((g - g|[p]))[q^2]$$

Now we apply the operator $\tau_{p^2q^2}$. We claim that only the term $\epsilon_p(q)((g - g|[p]))[q^2]|_1\tau_{p^2q^2}$ will survive after considering the Rankin-Selberg convolution with $\tilde{f}_0$. Indeed as $\tilde{f}_0$ has no Euler factors at $m$ its Rankin-Selberg convolution with a form $g'$ with $a(n, g') = 0$ if $(n, q) = 1$, will be trivial. Consider now, $g|[q^ip^j]|\tau_{p^2q^2}$ with $i = 0, 1, 2$ and $j = 0, 1$. Then,

$$g|[q^ip^j]|\tau_{p^2q^2} = \frac{1}{(q^ip^j)^{\frac{1}{2}}} g|_1 \begin{pmatrix} q^ip^j & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ p^2q^2 & 0 \end{pmatrix} =$$

$$= \frac{1}{(q^ip^j)^{\frac{1}{2}}} g|_1 \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix} \begin{pmatrix} pq^2 & 0 \\ 0 & q^ip^j \end{pmatrix} = \frac{1}{(q^ip^j)^{\frac{1}{2}}} W(g)g|_1 \begin{pmatrix} q^ip^j & 0 \\ 0 & q^ip^j \end{pmatrix} \begin{pmatrix} p^{1-j}q^{2-i} & 0 \\ 0 & 1 \end{pmatrix} =$$

$$= \frac{1}{(q^ip^j)^{\frac{1}{2}}} W(g)g|_1 \begin{pmatrix} p^{1-j}q^{2-i} & 0 \\ 0 & 1 \end{pmatrix} = (\frac{p^{1-j}q^{2-i}}{(q^ip^j)})^{\frac{1}{2}} W(g)g|[p^{1-j}q^{2-i}]$$

So we see that if $i \neq 2$ then $D(1, \tilde{f}_0, g|[q^ip^j]|\tau_{p^2q^2}) = 0$. Moreover we see from the above computations that the term $\epsilon_p(q)((g - g|[p]))[q^2]$ equals $\frac{\epsilon_p(q)}{q}p^{-\frac{1}{2}}W(g)(g - pg|[p])$, which concludes our claim. Now it is not hard to check that our argument extends to the general case. One has again to observe that only terms of the form $g'|[m^2]$ will survive after the Rankin-Selberg convolution. ■

## 5.2 The case $p > 3$

Let us fix some notation first. Let us write $K := \mathbb{Q}(\mu_p)$ and $F := \mathbb{Q}(\mu_p)^+$, a totally real field and $[K : F] = 2$. Moreover we write $\mathfrak{p}$ for the unique prime above $p$ in $F$ and $\epsilon_{\mathfrak{p}}$ for the non-trivial character of $K/F$. Let us also denote by $\chi$ a non-trivial character of the cyclic extension $K(\sqrt[p]{m})/K$ of degree $p$ for some $p^{th}$ power free integer $m$. Moreover we write $\rho := Ind_K^F(\chi)$ and $R := Ind_F^{\mathbb{Q}}(\rho) = Ind_K^{\mathbb{Q}}(\chi)$. Also we write $\sigma := Ind_K^F(1) = 1 \oplus \epsilon_{\mathfrak{p}}$ and $\Sigma = Ind_K^{\mathbb{Q}}(1) = \oplus_{r=1}^{p} \theta^r$ for some character $\theta$ of $Gal(K/\mathbb{Q})$. Our aim is to establish congruences between the quantities

$$Q(R) := e_p(R) u^{-v_p(N_R)} \frac{P_p(R, u^{-1})}{P_p(R, w^{-1})} \frac{L_S(E/\mathbb{Q}, R, 1)}{(\Omega(E)_+ \Omega(E)_-)^{\frac{p-1}{2}}}$$

$$Q(\Sigma) := e_p(\Sigma) u^{-v_p(N_\Sigma)} \frac{P_p(\Sigma, u^{-1})}{P_p(\Sigma, w^{-1})} \frac{L_S(E/\mathbb{Q}, \Sigma, 1)}{(\Omega(E)_+ \Omega(E)_-)^{\frac{p-1}{2}}}$$

where $S$ is the set of primes consisting of $p$ and $q|m$. From the inductive properties of the $L$ functions we note the equalities $L_S(E/\mathbb{Q}, R, 1) = L_S(E/\mathbb{Q}, Ind_K^{\mathbb{Q}}(\chi), 1) = L_S(E/\mathbb{Q}, Ind_K^F Ind_F^{\mathbb{Q}}(\chi), 1) = L_S(E/F, Ind_K^F(\chi), 1) = L_S((E/F, \rho, 1)$ and also in the same way $L_S(E/\mathbb{Q}, \Sigma, 1) = L_S(E/F, \sigma, 1)$. Of course here the set $S$ contains the primes of $\mathcal{O}_F$ and $\mathcal{O}_K$ above $m$ and of course $\mathfrak{p}$. Moreover as the inductive properties hold for Euler factors we can also conclude that $P_p(R, X) = P_{\mathfrak{p}}(\rho, X)$ and $P_p(\Sigma, X) = P_{\mathfrak{p}}(\sigma, X)$. For the local epsilon factors and the conductor we know that they are inductive in degree zero and so we have that, $\frac{e_p(R)}{e_p(\Sigma)} = \frac{e_{\mathfrak{p}}(\rho)}{e_{\mathfrak{p}}(\sigma)}$. We now consider the quantities,

$$Q(\rho) := e_{\mathfrak{p}}(\rho) u^{-v_{\mathfrak{p}}(N_\rho)} \frac{P_{\mathfrak{p}}(\rho, u^{-1})}{P_{\mathfrak{p}}(\rho, w^{-1})} \frac{L_S(E/F, \rho, 1)}{(\Omega(E)_+ \Omega(E)_-)^{\frac{p-1}{2}}}$$

$$Q(\sigma) := e_{\mathfrak{p}}(\sigma) u^{-v_{\mathfrak{p}}(N_\sigma)} \frac{P_{\mathfrak{p}}(\sigma, u^{-1})}{P_{\mathfrak{p}}(\sigma, w^{-1})} \frac{L_S(E/F, \sigma, 1)}{(\Omega(E)_+ \Omega(E)_-)^{\frac{p-1}{2}}}$$

Let us now write $f \in S_2(\Gamma_0(N), \mathbb{Z})$ for the primitive cusp form that we can associate to $E$ and by $\pi_f$ the corresponding cuspidal automorphic representation, i.e. $L(E/\mathbb{Q}, s) = L(f, s) = L(\pi_f, s)$. Notice that as $F/\mathbb{Q}$ is cyclic we can consider the base change of $\pi_f$ from $\mathbb{Q}$ to $F$, a cuspidal automorphic representation $\pi_\phi$ of $GL(2, \mathbb{A}_F)$ such that $L(\pi_\phi, s) = \prod_{r=1}^{\frac{p-1}{2}} L(\pi_f \otimes \eta^r, s)$ for a finite order Hecke character $\eta$ that corresponds to a Galois character that generates $Gal(F/\mathbb{Q})^\vee$. Let us write $\phi$ for the Hilbert modular form of parallel weight two that we attach to $\pi_\phi$ in the canonical way that we have described in the introduction.

By automorphic induction for degree two extensions we can associate a Hilbert modular form $\mathbf{g}_\rho \in S_1(N_{K/F}(\mathbf{f}_\chi)\mathfrak{p}, \epsilon_{\mathfrak{p}})$ to $\rho$ (the character is just $\epsilon_{\mathfrak{p}}$ as $\chi$ is anti-cyclotomic), and an Eisenstein series $\mathbf{E}_\sigma \in M_1(\mathfrak{p}, \epsilon_{\mathfrak{p}})$ to $\sigma$. From now on we will write $M$ for the ideal $N_{K/F}(\mathbf{f}_\chi)$ of $\mathcal{O}_F$. Note that as we assume that $E$ is semi-stable we have that $L(E/F, \chi, 1) = D(\phi, g_\rho, 1)$ and $L(E/F, \sigma, 1) = D(\phi, \mathbf{E}_\sigma, 1)$. Moreover we write $\phi_0$ for the "ordinary" Hilbert modular form that one can attach to $\phi$ such that $C(\mathfrak{q},\phi_0) = C(\mathfrak{q},\phi)$ if $\mathfrak{q} \neq \mathfrak{p}$ and $C(\mathfrak{p},\phi_0)$ is the $p$-adic unit root of the equation $x^2 - C(\mathfrak{p},\phi) + p = 0$. Note that $C(\mathfrak{p},\phi) = a(p,f)$. Finally by $\tilde{\phi}_0$ we denote the Hilbert modular form that we obtain from $\phi_0$ by removing the Euler factors above $m$.

We can extend all that we did above for the case of $p = 3$ to the more general setting of $p > 3$, where now instead of working with elliptic modular forms we work with Hilbert modular forms. In particular we could introduce the notion of a $p$-adic Hilbert modular form as in Hida [15] using the $q$-expansion principle or their moduli interpretation as in Katz [22]. However as we said in the introduction we do not have yet a theorem for the general case for reasons that will explain later. So in this section we restrict ourselves to just state the following theorem, a proof of which can be found in [2]. It will be enough in order to address the issues that prevent us from proving a general theorem for $p > 3$.

**Theorem 3** *Let $\gamma$ be the power of $\mathfrak{p}$ in the level of $\boldsymbol{E}_\sigma$ and $\beta$ the power of $\mathfrak{p}$ in the level of $\boldsymbol{g}_\rho$. Consider the quantities,*

$$Q(\rho) := C(\mathfrak{p},\phi_0)^{-(\beta-1)} p^{\frac{\beta}{2}} \frac{D(\tilde{\phi}_0, \boldsymbol{g}_\rho|_{\imath_\mathfrak{p}}|\tau_{M\mathfrak{p}^\beta}, 1)}{i^{\frac{p-1}{2}} \pi^{p-1} < \tilde{\phi}_0|\tau_{L\mathfrak{p}}, \tilde{\phi}_0 >_{L\mathfrak{p}}}$$

$$Q(\sigma) := C(\mathfrak{p},\phi_0)^{-(\gamma-1)} p^{\frac{\gamma}{2}} \frac{D(\tilde{\phi}_0, \boldsymbol{E}_\sigma|_{\imath_{M\mathfrak{p}}}|\tau_{M\mathfrak{p}^\gamma}, 1)}{i^{\frac{p-1}{2}} \pi^{p-1} < \tilde{\phi}_0|\tau_{L\mathfrak{p}}, \tilde{\phi}_0 >_{L\mathfrak{p}}}$$

*Then there exists a well-determined constant $c(\phi,m) \in \mathcal{O}_\mathfrak{p}$ depending only on $\phi$ and $m$ such that, both $c(\phi,m)Q(\rho)$ and $c(\phi,m)Q(\sigma)$ are $p$-adically integral and,*

$$c(\phi,m)Q(\rho) \equiv c(\phi,m)Q(\sigma) \mod \mathfrak{p}$$

# 6 Congruences between special values

In this section our aim is to obtain a better understanding of the nature of the constant $c(f,m)$ appearing in theorem 2 and its relation with the choice of our periods. Vaguely speaking, we show that the reason that this constant appears is that the Petersson inner product is not the right choice to obtain integral values of the ratio (L-values)/(automorphic periods) and this constant measures this failure.

We start by showing that our map $h_2(\Gamma_0(Npm^2);\mathbb{Z}_p) \to \mathbb{Z}_p$ given by $T(n) \mapsto a(n,\tilde{f}_0)$, under the assumptions stated in the introduction, induces a decomposition of the form, $h_2(\Gamma_0(Npm^2);\mathbb{Q}_p) = \mathbb{Q}_p \times A$. This will be done by showing that actually factors through a local ring of $h_2(\Gamma_0(Npm^2);\mathbb{Z}_p)$ that is reduced. Then our next goal is to relate the quantity $c(f,m)$ to the periods $< \tilde{f}_0|\tau_{N_\Sigma}, \tilde{f}_0 >$, with $N_\Sigma = N_E m^2 p$ that appear in the congruences of theorem 2. Namely we will show that there is a period determinant, we call it $\Omega(f)_\Sigma$ for $\Sigma$ the set of primes dividing $m$, such that $c(f,m)\Omega(f)_\Sigma =< \tilde{f}_0|\tau_{N_\Sigma}, \tilde{f}_0 >$, up to a $p$-adic unit.

In order to conclude the theorem we will show that actually the quantity $\Omega(f)_\Sigma$ is independent of $m$ and the prime $p$. Hence we are reduced down to the primitive form $f$ and the existence of a strong parametrization of $E$ by $X_0(N)$ allow us to obtain a relation with the Néron periods.

As we mentioned in the introduction, in this section we rely on Wiles' deep results in [28]. Indeed the local ring through which our map factors is nothing else than the reduced algebra $\mathbb{T}_\Sigma$, using Wiles' notation, that he eventually proves to be isomorphic to some universal deformation rings that classify Galois representations with some predetermined properties that deform a fixed modular mod $p$ representation. Actually for our purposes we need the minimal level i.e. $\Sigma = \emptyset$, to be the conductor of the elliptic curve and this is the reason for imposing the assumption that the level of the elliptic curve is the same with the conductor of the reduced (mod 3) representation.

Then we define the module of congruences using this reduced local ring. It measures congruences between our modified form $\tilde{f}_0$ and other normalized eigenforms. The annihilator of this module will be eventually the quantity $c(f,m)$. We will compare this module of congruences with what may be called the cohomological module of congruences. Under our assumptions, Wiles' results on the freeness of $H^1(X_0(N_\Sigma), \mathbb{Z}_p)_\mathfrak{m}$ as a $h_2(\Gamma_0(Npm^2); \mathbb{Z}_p)_\mathfrak{m}$-module, where here we write $\mathfrak{m}$ for the maximal ideal that corresponds to the form that one obtains reducing $\tilde{f}_0$ modulo $p$, will give that actually the two modules are isomorphic. This will relate $c(f,m)$ to the periods $< \tilde{f}_0|\tau_{N_\Sigma}, \tilde{f}_0 >$. Then again a deep result of Wiles, the generalization of the so-called "Ihara's lemma" will essentially say that this relation does not depend on the change of level that we have introduced to $f$ by removing the Euler factors at $m$ and modifying the one at $p$. Hence we can reduce our study to the initial level $N$ where the modularity of the elliptic curve provides us the way to obtain the relation with Néron periods.

We would like here to mention that in this section we have benefited the most from the article of Darmon, Diamond and Taylor [6] based on Wiles' fundamental paper [28]. Most of the constructions and proofs here are minor modifications, mainly just restricting their constructions to our specific case, of the ones done in their paper.

**Structure of Hecke algebras:** In this section we collect some well known facts about the structure of the integral Hecke algebra $h_2(\Gamma_0(N); \mathbb{Z})$ and its completion $h_2(\Gamma_0(N); \mathbb{Z}_p)$ at some prime $p$. Our main reference is [6]. We fix a finite extension $K$ of $\mathbb{Q}_p$ and we denote by $\mathcal{O}_K$ the ring of integers. We write $\lambda$ for the maximal ideal in $\mathcal{O}_K$ and $k$ for $\mathcal{O}/\lambda$. Let us moreover fix algebraic closures $\overline{K}$ and $\overline{k}$ of $K$ and $k$. We consider the Hecke algebras (a) $h_2(\Gamma_0(N); \mathcal{O}_K) = h_2(\Gamma_0(N); \mathbb{Z}) \otimes_\mathbb{Z} \mathcal{O}_K$, (b) $h_2(\Gamma_0(N); K) = h_2(\Gamma_0(N); \mathcal{O}_K) \otimes_{\mathcal{O}_K} K$ and (c) $h_2(\Gamma_0(N); k) = h_2(\Gamma_0(N); \mathcal{O}_K) \otimes_{\mathcal{O}_K} k$. From the going-up and going-down theorems we have that the maximal prime ideals $\mathfrak{m} \subset h_2(\Gamma_0(N); \mathcal{O}_K)$ are above the prime $\lambda$ i.e. $\mathfrak{m} \cap \mathcal{O}_K = (\lambda)$ and for the minimal primes $\mathfrak{p}$, $\mathfrak{p} \cap \mathcal{O}_K = (0)$. Moreover we have the isomorphism ([6], p.90) $h_2(\Gamma_0(N); \mathcal{O}_K) \xrightarrow{\sim} \prod_\mathfrak{m} h_2(\Gamma_0(N); \mathcal{O}_K)_\mathfrak{m}$ where the product is over the finitely many maximal ideals of $h_2(\Gamma_0(N); \mathcal{O}_K)$.

Let $f \in S_2(\Gamma_0(N); \overline{K})$ be a normalized eigenform and consider the K-algebra homomorphism $\lambda_f : h_2(\Gamma_0(N); K) \to \overline{K}$, that sends $T(n) \mapsto a(n,f)$. Using this we can associate with $f$ a maximal ideal of $h_2(\Gamma_0(N); K)$ by $\ker(\lambda_f)$ which depends only on the $G_K$ conjugacy class of $f$. In the same way we can associate a maximal ideal of $h_2(\Gamma_0(N); k)$ to a normalized eigenform $g \in S_2(\Gamma_0(N); \overline{k})$. We usually write $\overline{f}$ for the reduction modulo $\lambda$ of a form $f$ with integral Fourier expansion. The following proposition is taken from [6], p. 90.

**Proposition 3** *Let us denote by $S_2(N;\overline{K})(G_K)$ the normalized eigenforms in $S_2(\Gamma_0(N);\overline{K})$ up to $G_K$ conjugacy and by $S_2(N;\overline{k})(G_k)$ the normalized eigenforms in $S_2(\Gamma_0(N);\overline{k})$ up to $G_k$ conjugacy. Then the elements in $S_2(N;\overline{K})(G_K)$ (resp in $S_2(N;\overline{k})(G_k)$) are in bijection with the maximal ideals of $h_2(\Gamma_0(N);K)$ (resp maximal ideals of $h_2(\Gamma_0(N);k)$) which in turn are in bijection with the minimal primes of $h_2(\Gamma_0(N);\mathcal{O}_K)$ (maximal primes of $h_2(\Gamma_0(N);\mathcal{O}_K)\}$).*

Finally we note that if we let $\mathfrak{m}$ be a maximal ideal in $h_2(\Gamma_0(N);\mathcal{O}_K)$ and consider the maximal ideals $\mathfrak{p} \subset h_2(\Gamma_0(N);K)$ with $\mathfrak{p} \cap h_2(\Gamma_0(N);\mathcal{O}_K) \subset \mathfrak{m}$ then we have an isomorphism $h_2(\Gamma_0(N);\mathcal{O}_K)_{\mathfrak{m}} \otimes_{\mathcal{O}_K} K \xrightarrow{\sim} \prod_{\mathfrak{p}} h_2(\Gamma_0(N);K)_{\mathfrak{p}}$. We also mention what the Atkin-Lehner theory implies for the Hecke ring $h_2(\Gamma_0(N);K)$ under the assumption that the field $K$ contains all the coefficients of all the primitive forms of conductor dividing $N$ and trivial character. Let us denote by $\mathcal{P}(N)$ the set of primitive forms of conductor dividing $N$. Then we have that, $S_2(\Gamma_0(N);K) = \oplus_{f\in\mathcal{P}(N)} S_{K,f}$ where $S_{K,f}$ is the $K$-linear span of $\{f(\alpha z) : \alpha \mid N/N_f\}$ with $N_f$ the conductor of the primitive form $f$. For each $f = \sum_{n\geq 0} a(n,f)q^n$ in $\mathcal{P}(N)$ we denote by $h_2(\Gamma_0(N);K)[f]$ the image of $h_2(\Gamma_0(N);K)$ in $End_K(S_{K,f})$. We consider the polynomial ring, $A_{K,f} = K[u_{f,q} : \forall q \mid N/N_f]$ and the ideal $I_{K,f} \subset A_{K,f}$ generated by the polynomials $P_{f,q}(u_{f,q}) = u_{f,q}^{v_q(N/N_f)-1}(u_{f,q}^2 - a(q,f)u_{f,q} + 1_{(N_f)}q)$ where we write $v_q(N/N_f)$ for the valuation at $q$. We now prove the following, which is a version of lemma 4.4 of [6],

**Lemma 6** *There is an isomorphism of $K$-algebras $\phi : h_2(\Gamma_0(N);K) \xrightarrow{\sim} \prod_{f\in\mathfrak{N}} A_{K,f}/I_{K,f}$ defined by,*

$$\phi(T(q))_f := \begin{cases} a(q,f), & \textit{if } (q, N/N_f) = 1; \\ u_{f,q} \mod I_{K,f}, & \textit{otherwise.} \end{cases}$$

**Proof** We define the $K$-algebra homomorphism, $\Theta_f : A_{K,f} \to h_2(\Gamma_0(N);K)[f]$ by $u_{f,q} \mapsto T(q)$. Notice that the polynomial $P_{f,q}$ is the characteristic polynomial of the operator $T(q)$ acting on the space spanned by the forms $\{f(\alpha q^i z) : i = 1,\ldots,v_q(N/N_f)\}$ for each $\alpha$ dividing $N/N_f q^{v_q(N/N_f)}$. That implies that $I_{K,f}$ is in the kernel $\Theta_f$. Hence, we have the following surjection, which we denote by $\Theta$,

$$\Theta : \prod_f A_{K,f}/I_{K,f} \twoheadrightarrow \prod_f h_2(\Gamma_0(N);K)[f]$$

But since $h_2(\Gamma_0(N);K) \hookrightarrow \prod_f h_2(\Gamma_0(N);K)[f]$ the following counting argument establishes the isomorphism,

$$dim_K(h_2(\Gamma_0(N);K)) = dim_K S_2(\Gamma_0(N),K) = \sum_f \sigma_0(N/N_f) = \sum_f dim_K A_{K,f}/I_{K,f}$$

with $\sigma_0(n) = \sum_{0<d|n} 1$. ■

**Reduced Hecke algebras:** Recall that we are interested in the $K$-algebra homomorphism $h_2(\Gamma_0(Npm^2),K) \to K$ that corresponds to a normalized eigenform that arises from an ordinary primitive form $f$ of conductor $N$ after "removing" the Euler factors at the primes dividing $m$ and modifying the Euler factor at $p$ by keeping only the $p$-adic unit part. In this section we are going to show that under some assumptions on $f$, that we now describe, this homomorphism factors through a local ring $h_2(\Gamma_0(Nm^2p, \mathcal{O})_{\mathfrak{m}}$ that is reduced. We recall that if we write $\rho_f := \rho_{f,p} : G_{\mathbb{Q}} \to GL_2(\mathbb{Z}_p)$ for the $p$-adic representation attached to $f$, then we assume that the reduced, mod p, representation $\overline{\rho} := \rho \mod p$, is irreducible. Let us now write $N(\bar{\rho})$ for the Artin conductor of $\bar{\rho}$. Then we also assume that $N(\bar{\rho}) = N$. Finally $f$ is the normalized primitive form corresponding to an elliptic curve $E$, which we assume has good ordinary reduction at $p$.

As we have already mentioned in the introduction we will show that our local ring $h_2(\Gamma_0(Npm^2), \mathcal{O})_{\mathfrak{m}}$ is isomorphic to some reduced ring $\mathbb{T}_{\Sigma}$ that appear in Wiles' work. For the purposes of this section we do not really need to establish this identification as we can work only with the full Hecke algebra. However in order to make some remarks when we later consider the case $p > 3$, of course in a Hilbert modular form setting, we will refer to this identification.

Let us start by fixing a general setting for this section. We write $f$ for a normalized primitive form of conductor $N$ and trivial character that is ordinary at $p$. We will write $\rho := \rho_f$ for the $p$-adic representation that we attach to $f$, and $\bar{\rho} := \bar{\rho}_f$ for its mod $p$ representation. From now on we assume that the local field $K$ is always sufficiently large, in the sense that always contains all the Fourier coefficients of the cusp forms that we consider. We fix a set $\Sigma$ of primes $\ell \neq p$ that do not divide $N$. We define, $N_{\Sigma} := Np\prod_{\ell \in \Sigma} \ell^2$. Moreover we write $\mathcal{N}_{\Sigma}$ for the set of primitive forms $g$ of conductor dividing $N_{\Sigma}$ and trivial character with the property that,

$$a(q,g) \mod \lambda' = tr(\overline{\rho}(Frob_q)) \quad \forall q \quad with \quad (q, N_{\Sigma}) = 1$$

where $\lambda'$ is the maximal ideal in the field $K_g$, the minimal $\mathbb{Q}_p$ extension that contains the coefficients of $g$. Notice that if we write $\rho_g$ for the $p$-adic representation that we can attach to the newform $g$ then the above condition gives that $\overline{\rho} \otimes_k k_g \cong \overline{\rho}_g$ where $\overline{\rho}_g$ is the unique, up to isomorphism, mod $p$ semi-simple representation such that $tr(\overline{\rho}_g(Frob_q)) = a(q,g) \mod \lambda'$ for all $(q, N(\overline{\rho}_g)p) = 1$. Let $g \in \mathcal{N}_{\Sigma}$ be any of our selected primitive forms and for any such $g$ we consider the normalized eigenform $g' \in S_2(\Gamma_0(N_{\Sigma}))$ that is defined by,

1. $a(q,g') = a(q,g)$ if $q$ does not divide $N_{\Sigma}/N_g$
2. $a(q,g') = 0$ if $q \neq p$ and $q$ divides $N_{\Sigma}/N_g$
3. $a(p,g') := u(g)$, the $p$-adic unit root of the equation $X^2 - a(p,g)X + p = 0$ if $p$ divides $N_{\Sigma}/N_g$. Note that this makes sense as $\bar{\rho}$ comes from an elliptic curve with good ordinary reduction at $p$.

It follows from lemma 4.6 of [6] that the form $\overline{g'} \in S_2(\Gamma_0(N_{\Sigma}); k)$ is independent of $g$, and more precisely it is characterized by the conditions (1) $a(q, \overline{g'}) = tr\overline{\rho}_{I_q}(Frob_q)$ if $q = p$ or $q \notin \Sigma$, where we write $\overline{\rho}_{I_q}$ for the coinvariant space of the inertia at $q$, and (2) $a(q, \overline{g'}) = 0$ if $q \in \Sigma$. We then write $\mathfrak{m}$ for the maximal ideal in $h_2(\Gamma_0(N_\Sigma);\mathcal{O})$ that corresponds to $\overline{g'}$ by proposition 3. Then we claim that $h_2(\Gamma_0(N_\Sigma);\mathcal{O})_{\mathfrak{m}} \otimes_{\mathcal{O}} K$ is a semi-simple $K$-algebra. Indeed we have that,

**Proposition 4** *There is a $K$-algebra isomorphism,*

$$\phi : h_2(\Gamma_0(N_\Sigma);\mathcal{O})_{\mathfrak{m}} \otimes_{\mathcal{O}} K \xrightarrow{\sim} \prod_{g\in\mathcal{N}_\Sigma} K$$

*given by*

$$(\phi(T_q))_g = \begin{cases} a(q,g), & \text{if } q \notin \Sigma \cup \{p\}; \\ 0, & \text{if } q \in \Sigma; \\ u(g), & \text{if } q = p. \end{cases}$$

**Proof** By lemma 6 we have an isomorphism

$$h_2(\Gamma_0(N_\Sigma);K) \xrightarrow{\sim} \prod_{g\in\mathcal{P}(N_\Sigma)} A_{K,g}/I_{K,g}$$

where $\mathcal{P}(N_\Sigma)$ is the set of primitive forms in $S_2(\Gamma_0(N_\Sigma);K)$. Recall also that for a maximal ideal $\mathfrak{m} \subset h_2(\Gamma_0(N_\Sigma);\mathcal{O}_K)$ we have that

$$h_2(\Gamma_0(N_\Sigma);\mathcal{O}_K)_{\mathfrak{m}} \otimes_{\mathcal{O}_K} K \xrightarrow{\sim} \prod_{\mathfrak{p}} h_2(\Gamma_0(N_\Sigma);K)_{\mathfrak{p}}$$

for all prime ideals $\mathfrak{p} \subset h_2(\Gamma_0(N_\Sigma);K)$ that restricted to $h_2(\Gamma_0(N_\Sigma);\mathcal{O}_K)$ they are contained in $\mathfrak{m}$. Hence we obtain the following isomorphism,

$$h_2(\Gamma_0(N_\Sigma);\mathcal{O}_K)_{\mathfrak{m}} \otimes_{\mathcal{O}_K} K \xrightarrow{\sim} \prod_{g\in\mathcal{P}(N_\Sigma)} \prod_{\mathfrak{p}\in\mathcal{M}_g} (A_{K,g}/I_{K,g})_{\mathfrak{p}}$$

where $\mathcal{M}_g$ is the set of prime ideals in $A_{K,g}/I_{K,g}$ that their image under the map $\Theta_g$ (with notation as in lemma 6) when restricted to $h_2(\Gamma_0(N_\Sigma);\mathcal{O}_K)$ is in $\mathfrak{m}$. But if $g$ is not in $\mathcal{N}_\Sigma$ then $\mathcal{M}_g$ is empty. If $g$ is in $\mathcal{N}_\Sigma$ then there is a unique prime ideal in $\mathcal{M}_g$, call it $\mathfrak{p}_{g'}$, that restricts inside $\mathfrak{m}$. Indeed, it is the prime ideal that corresponds to the normalized eigenform $g'$ constructed above as we have shown that it is the unique normalized eigenform in $S_{K,g}$ with the required reduction. This prime ideal is actually the kernel of the map $A_{K,g}/I_{K,g} \to K$ sending $u_{f,g} \mapsto a(q,g')$, and after localizing we obtain $(A_{K,g}/I_{K,g})_{\mathfrak{p}_{g'}} \xrightarrow{\sim} K$. Finally the explicit description of the isomorphism in lemma 6 gives the description of the isomorphism $\phi$. ■

Now we are going to introduce the algebras $\mathbb{T}_\Sigma$ that appear in Wiles' work. As we mentioned, we will not make any direct use of them in the case of $p = 3$. We consider the $\mathcal{O}_K$-algebra $\mathbb{T}'_\Sigma := \prod_{g\in\mathcal{N}_\Sigma} \mathcal{O}_{K_g}$. and we define the $\mathcal{O}_K$-subalgebra $\mathbb{T}_\Sigma \subset \mathbb{T}'_\Sigma$ generated over $\mathcal{O}_K$ by the elements $T(q) := (a(q,g))_g$ for all $q$ relatively prime to $N_\Sigma$. Note that $\mathbb{T}_\Sigma$ is reduced as we consider only the "good", i.e. away from the level, Hecke eigenvalues. In Wiles' work this algebra is shown to be a deformation ring of the representation $\bar{\rho}$. One of the crucial steps in his work is that he identifies this algebra with a localized part of the full Hecke algebra. In our case it follows from Proposition 4.7 in [6] that there is an isomorphism of $\mathcal{O}_K$-algebras, $\phi : h_2(\Gamma_0(N_\Sigma);\mathcal{O}_K)_{\mathfrak{m}} \xrightarrow{\sim} \mathbb{T}_\Sigma$ given by $T_q \mapsto T(q)$ for all $q$ relatively prime to $N_\Sigma$.

**Modules of congruences** For our primitive form $f \in S_2(\Gamma_0(N_f;\mathbb{Z}_p)$ we are interested in the map $\pi_\Sigma : h_2(\Gamma_0(N_\Sigma);\mathbb{Z}_p) \to \mathbb{Z}_p$ induced from the normalized eigenform $f' = \tilde{f}_0 = f_0|\imath_m$, with $m = \prod_{\ell\in\Sigma} \ell$. By the universal property of localization, this map factors as,

$$\pi_\Sigma : h_2(\Gamma_0(N_\Sigma);\mathbb{Z}_p) \to h_2(\Gamma_0(N_\Sigma);\mathbb{Z}_p)_\mathfrak{m} \to \mathbb{Z}_p$$

where the maximal ideal $\mathfrak{m}$ is as in the previous section. If we use the identification $h_2(\Gamma_0(N_\Sigma);\mathbb{Z}_p)_\mathfrak{m} \xrightarrow{\sim} \mathbb{T}_\Sigma$ then the map can be realized as the projection to the component corresponding to $f$. Moreover we have shown that $h_2(\Gamma_0(N_\Sigma);\mathbb{Z}_p)_\mathfrak{m} \cong \mathbb{T}_\Sigma$ is reduced and in particular the map $\pi_\Sigma$ induces a splitting $h_2(\Gamma_0(N_\Sigma);\mathbb{Z}_p)_\mathfrak{m} \otimes_\mathbb{Z} \mathbb{Q}_p = \mathbb{Q}_p \times A$, where we write just $\mathbb{Q}_p$ as $f$ has rational coefficients. Recall that we write $1_{\mathbb{Q}_p}$ for the idempotent corresponding to the copy of $\mathbb{Q}_p$. We would like to study its "denominator" i.e. a quantity $c(f,m)$ such that $c(f,m)1_{\mathbb{Q}_p}$ is integral. For this we now introduce the notion of the module of congruences.

We start with some general definitions and properties of the module of congruences as for example are given by Hida in his book, see [17] (page 276). Let us write $h$ for a local ring $h_2(\Gamma_0(N),\mathcal{O}_K)_\mathfrak{m}$ for some $N$. Moreover let us assume that $h$ is reduced and that we are given a map $\phi : h \to \mathcal{O}_K$, such that it induces an $K$-algebra decomposition,

$$h \otimes_{\mathcal{O}_K} K \cong K \times A$$

for some $K$-algebra $A$. Let us denote by $1_\phi$ the idempotent that corresponds to the first summand $K$. We define $\mathfrak{a} := Ker(h \to A)$ and $\wp := Ker(\phi)$. Note also that $Ann_h(\wp) = \mathfrak{a}$.

**Definition 6** *The module of congruences $C_0(h)$ of $\phi : h \to \mathcal{O}_K$ is defined as,*

$$C_0(h) := (h/\mathfrak{a}) \otimes_{h,\phi} \mathcal{O}_K \cong \frac{h}{\mathfrak{a}\oplus\wp} \cong \mathcal{O}_K/\phi(\mathfrak{a}) \cong 1_\phi h/\mathfrak{a}$$

We now consider the module of congruences $C_0(h)$ for our reduced ring $h := h_2(\Gamma_0(N_\Sigma),\mathbb{Z}_p)_\mathfrak{m}$ and our map $\pi_\Sigma$. We will compare it with a "cohomological" module of congruences, following the terminology of Hida and Ribet, which we will introduce below. Let us write $X$ for the compact modular curve $X_0(N_\Sigma)$. We consider the first cohomology group $H^1(X,\mathbb{Z}_p)$ and we have seen in chapter three that this as a Hecke module over $h_2(\Gamma_0(N_\Sigma);\mathbb{Z}_p)$. Moreover we consider the standard skew-symmetric bilinear perfect pairing as for example in [6] page 106,

$$(\cdot,\cdot) : H^1(X;\mathbb{Z}_p) \times H^1(X;\mathbb{Z}_p) \to \mathbb{Z}_p$$

Let us define $L := H^1(X;\mathbb{Z}_p)_\mathfrak{m}$, where we have localized $H^1(X;\mathbb{Z}_p)$ at the maximal ideal $\mathfrak{m}$. This is then an $h$-module. Let us consider the action of complex conjugation on $H^1(X;\mathbb{Z}_p)$ and define $L[+], L[-]$ for the eigenspaces of $L$. We define a cohomological module of congruences by,

$$C^{coh}(L[+]) := 1_{\mathbb{Q}_p}L[+]/1_{\mathbb{Q}_p}L[+] \cap L[+] \cong L[+]^{\mathbb{Q}_p}/L[+]_{\mathbb{Q}_p} \cong \frac{L[+]}{L[+][\wp]\oplus L[+][\mathfrak{a}]}$$

where we have set $L[+]^{\mathbb{Q}_p} := 1_{\mathbb{Q}_p} L[+]$, the projection of $L[+]$ to the first component of the decomposition $L[+] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = 1_{\mathbb{Q}_p}(L[+] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \oplus (1 - 1_{\mathbb{Q}_p})(L[+] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ induced by the splitting of the Hecke algebra. Also $L[+]_{\mathbb{Q}_p} := 1_{\mathbb{Q}_p} L[+] \cap L[+] = L[+][\wp]$ with $\wp$ and $\mathfrak{a}$ as in the definition of the module of congruences, the restriction of $L[+]$ to the first component. Here, as usual, $L[+][\wp] = \{l \in L[+] : \lambda(l) = 0, \ \forall \lambda \in \wp\}$. Note that $L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = H^1(X; \mathbb{Z}_p)_{\mathfrak{m}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is free of rank two over $h \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and hence $L[\pm] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ of rank one. In particular we have that $L[+]_{\mathbb{Q}_p}$ is a free $\mathbb{Z}_p$-module of rank one. We fix a basis $x_+$. The same holds for $L[-]_{\mathbb{Q}_p}$ and we fix a basis $x_-$.

**Lemma 7** *For the "cohomological" module of congruences we have,*

$$C^{coh}(L[+]) \cong \mathbb{Z}_p/((x_+, x_-))$$

**Proof** (See also [6] p. 105 and [17] p. 275) It is enough to show that $L[+]_{\mathbb{Q}_p} \cong Hom_{\mathbb{Z}_p}(L[-]^{\mathbb{Q}_p}, \mathbb{Z}_p)$ and $L[+]^{\mathbb{Q}_p} \cong Hom_{\mathbb{Z}_p}(L[-]_{\mathbb{Q}_p}, \mathbb{Z}_p)$. This follows by the prefect pairing $(\cdot, \cdot)$ on $L$. Indeed first we note that complex conjugation acts as $(a, b^\rho) = -(a^\rho, b)$, which explains the eigenspaces. Now let us show that $L[+]^{\mathbb{Q}_p} \cong Hom_{\mathbb{Z}_p}(L[-]_{\mathbb{Q}_p}, \mathbb{Z}_p)$ as the other claim is obtained similarly. Note that if we consider a basis $\{x_1, x_2, \cdots\}$ of $L$ as a $\mathbb{Z}_p$ module such that $x_1 = x_-$ then as $L$ is self-dual with respect to $(\cdot, \cdot)$ there is a dual basis $\{x_-^*, x_2^*, \cdots\}$ in $L$. Taking the projection $1_{\mathbb{Q}_p} x_-^*[+]$ gives a dual basis of $L[-]_{\mathbb{Q}_p}$. $\blacksquare$

We would like to compare the module of congruences $C_0(h)$ and $C^{coh}(L[+])$. Under our assumptions on $f$, i.e. it is $p$-ordinary and its modulo $p$ representation is irreducible, we have the following important theorem of Wiles [28].

**Theorem 4** (*Wiles*) *The $h$-module $H^1(X, \mathbb{Z}_p)_{\mathfrak{m}}$ is free (of rank two).*

We can conclude,

**Corollary 1** $C_0(h) \cong \frac{h}{\wp \oplus \mathfrak{a}} \cong \frac{1_{\mathbb{Q}_p} h}{\mathfrak{a}} \cong L[+]^{\mathbb{Q}_p}/L[+]_{\mathbb{Q}_p} \cong C^{coh}(L[+]) \cong \mathbb{Z}_p/((x_+, x_-))$.

Hence the quantity $(x_+, x_-) \in \mathbb{Z}_p$ annihilates the module of congruences and in particular we know that $(x_+, x_-) 1_{\mathbb{Q}_p} \in h \subset h_2(\Gamma_0(N_\Sigma), \mathbb{Z}_p)$. Hence we can define, up to $p$-adic units, $c(f, m) := (x_+, x_-)$. In the next section we will study the relation of $c(f, m)$ with the periods $< \tilde{f}_0 | \tau_{N_\Sigma}, \tilde{f}_0 >$ that appear in the first form of our congruences and eventually relate it to the Néron periods $\Omega_+(E)$ and $\Omega_-(E)$.

**Relations between periods of different levels:** The main aim now is to understand the relation between the quantity $c(f, m)$ and the automorphic periods $< \tilde{f}_0 | \tau_{N_\Sigma}, \tilde{f}_0 >$ that appear in our congruences in theorem 2. Recall that we are considering the homomorphism $\pi_\Sigma : h_2(N_\Sigma; \mathbb{Z}_p)_{\mathfrak{m}} \to \mathbb{Z}_p$ corresponding to our normalized eigenform $g := \tilde{f}_0$, arising from the newform $f$ (i.e. $g = f_0 | \imath_m$). Let us write $\mathbb{Z}_{(p)}$ for the localization of $\mathbb{Z}$ at $p$. Recall that we write $\wp$ for the kernel of $\pi_\Sigma$. Then we have the inclusion,

$$H^1(X_0(N_\Sigma); \mathbb{Z}_{(p)})[g] \subset H^1(X_0(N_\Sigma); \mathbb{Z}_p)_{\mathfrak{m}}[\wp] = L[\wp]$$

Let us choose a basis $\{x_+, x_-\}$ for $L[\wp]$ which is in the image of $H^1(X_0(N_\Sigma); \mathbb{Z}_{(p)})[g]$. We remind the reader that $p = 3$ and so as we are interested in statements up to $p$-adic units we can keep working with eigenspaces. We consider the $\mathbb{C}$ vector space $H^1(X_0(N_\Sigma); \mathbb{C})[g]$. The classical Eichler-Shimura isomorphism gives,

$$S_2(\Gamma_0; \mathbb{C}) \oplus \overline{S_2}(\Gamma_0; \mathbb{C}) \stackrel{\sim}{\to} H^1(X_0(N_\Sigma); \mathbb{C})$$

where we write $\overline{S_2}(\Gamma_0(N_\Sigma); \mathbb{C})$ for the space of the anti-holomorphic cusp forms. A canonical basis of $H^1(X_0(N_\Sigma); \mathbb{C})[g]$ is given by, $\{\omega_g, \overline{\omega_{g^\rho}}\}$ where $\omega_g = \sum a(n,g) q^{n-1} dq$ is a holomorphic differential on $X$ and $\overline{\omega}_{g^\rho} = \sum a(n,g) \bar{q}^{n-1} d\bar{q}$ an anti-holomorphic. Note that actually in the case of interest $g$ has rational coefficients, hence $g^\rho = g$. We now define a period $\Omega(f)_\Sigma$ as follows. We let $A_\Sigma$ be the two by two invertible matrix in $GL_2(\mathbb{C})$ such that $[\overline{\omega}_g, \omega_g] = [x_+, x_-] A_\Sigma$ and we define $\Omega(f)_\Sigma := det(A_\Sigma)$. Then,

**Lemma 8** *With notation as above we have the following equation,*

$$c(f,m)\Omega(f)_\Sigma = (x_+, x_-)\Omega(f)_\Sigma =< g|\tau_{N_\Sigma}, g >$$

**Proof** Just note that the skew-symmetry of the pairing for $A_\Sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ gives $(\overline{\omega}_g, \omega_g) = (ax_+ + cx_-, bx_+ + dx_-) = ad(x_+, x_-) - cb(x_+, x_-) = det(A_\Sigma)(x_+, x_-)$. But by the definition of the pairing we have that $(\overline{\omega}_g, \omega_g) = \int_X \overline{\omega}_{g|\tau_{N_\Sigma}} \wedge \omega_g =< g|\tau_{N_\Sigma}, g >$ ■

Hence in view of theorem 2, in order to conclude the congruences we need to relate the automorphic periods $\Omega(f)_\Sigma$ with the periods $\Omega_+(E)\Omega_-(E)$. The following theorem is taken from [6], p.108 and is based on Wiles' generalization of the so-called Ihara's lemma.

**Theorem 5** *We have, up to $p$-adic units, $\Omega(f)_\Sigma = \Omega(f)_{\Sigma=\emptyset} = \Omega(f)$ where $\Omega(f)$ is the period defined by taking $g = f$ above.*

Now we relate the period $\Omega(f)$ with the Néron periods $\Omega_+(E)\Omega_-(E)$, see [12], p. 255 and [28] p.537. Recall that we consider an elliptic curve $E/\mathbb{Q}$ of conductor $N$ and we write $f \in S_2(\Gamma_0(N); \mathbb{Z})$ for the primitive form of weight two and conductor $N$ associated to it. Let us fix a global minimal Weierstrass equation of $E$ over $\mathbb{Z}$ and let denote by $\omega_E$ the Néron differential of this equation. Let us consider the eigenspaces of $H_1(E(\mathbb{C}), \mathbb{Z})$ under the action of complex conjugation. We fix generators $\gamma_E^+$ and $\gamma_E^-$ for the spaces $H_1(E(\mathbb{C}); \mathbb{Z})^+$ and $H_1(E(\mathbb{C}); \mathbb{Z})^-$. Recall that we have defined the Néron periods as,

$$\Omega(E)_\pm := \int_{\gamma_E^\pm} \omega_E$$

If we write $\phi : X_0(N) \to E$ for the strong Weil parametrization of $E/\mathbb{Q}$ then we have that $\phi^*\omega_E = 2\pi\imath c_E f(z) dz$ where $c_E \in \mathbb{Q}^\times$ and in particular it has been proved by Mazur [25] that $c_E$ is a $p$-adic unit if $p^2 \nmid 4N$. By Poincaré duality we can pick a $\mathbb{Z}_{(p)}$ basis $\{c_1, c_2, \ldots, c_m\}$ of $H_1(X_0(N); \mathbb{Z}_{(p)})$ such that $\int_{c_j} \ell_i = \delta_{ij}$ for $i, j = \{1, 2\}$ and $\ell_1, \ell_2$ a basis of $H^1(X_0(N); \mathbb{Z}_{(p)})[\mathfrak{p}]$. Then we have,

$$det\begin{pmatrix} \int_{c_1}\omega_f & \int_{c_1}\bar{\omega}_f \\ \int_{c_2}\omega_f & \int_{c_2}\bar{\omega}_f \end{pmatrix} = det\begin{pmatrix} \int_{c_1}\ell_1 & \int_{c_1}\ell_2 \\ \int_{c_2}\ell_1 & \int_{c_2}\ell_2 \end{pmatrix}\Omega(f) = \Omega(f)$$

Moreover we have,

$$\left|det\begin{pmatrix} \int_{\phi(c_1)}\omega & \int_{\phi(c_1)}\bar{\omega} \\ \int_{\phi(c_2)}\omega & \int_{\phi(c_2)}\bar{\omega} \end{pmatrix}\right| = 4\pi^2 c_E^2 \left|det\begin{pmatrix} \int_{c_1}\omega_f & \int_{c_1}\bar{\omega}_f \\ \int_{c_2}\omega_f & \int_{c_2}\bar{\omega}_f \end{pmatrix}\right| = 4\pi^2 c_E^2\Omega(f)$$

And also,

$$\left|det\begin{pmatrix} \int_{\gamma_E^+}\omega & \int_{\gamma_E^+}\bar{\omega} \\ \int_{\gamma_E^-}\omega & \int_{\gamma_E^-}\bar{\omega} \end{pmatrix}\right| = 2|\Omega_+(E)\Omega_-(E)| = 2\Omega_+(E)\Omega_-(E)\imath^{-1}$$

where the last equality follows from the fact that we can pick $\gamma_E^{\pm}$ such that $\imath^{-1}\Omega_-(E)$ and $\Omega_+(E)$ are real positive. As

$$\left|det\begin{pmatrix} \int_{\gamma_E^+}\omega & \int_{\gamma_E^+}\bar{\omega} \\ \int_{\gamma_E^-}\omega & \int_{\gamma_E^-}\bar{\omega} \end{pmatrix}\right| = \left|det\begin{pmatrix} \int_{\phi(c_1)}\omega & \int_{\phi(c_1)}\bar{\omega} \\ \int_{\phi(c_2)}\omega & \int_{\phi(c_2)}\bar{\omega} \end{pmatrix}\right|$$

up to $p$-adic units, we have,

**Theorem 6** *The relation of the period $\Omega(f)$ with the periods $\Omega_+(E)$ and $\Omega_-(E)$, up to $p$-adic units, is given by the equation,*

$$\pi^2\imath\Omega(f) = \Omega_+(E)\Omega_-(E)$$

Putting all together, theorem 2, lemma 8, theorem 5 and the above theorem we conclude

**Theorem 7** *Consider an elliptic curve $E$ as in the introduction. Let $m$ be a power free positive integer with $(m, N_E) = (m,p) = 1$ with $p = 3$. Consider the Galois extension $\mathbb{Q}(\mu_p, \sqrt[p]{m})/\mathbb{Q}$ and let $\rho$ be the unique non-trivial two dimensional Artin-representation that factors through $Gal(\mathbb{Q}(\mu_p, \sqrt[p]{m})/\mathbb{Q})$. Then,*

$$e_p(\rho)u^{-v_p(N_\rho)}\frac{P_p(\rho,u^{-1})}{P_p(\rho,w^{-1})}\frac{L_{\{p,q|m\}}(E\otimes\rho,1)}{\Omega_+(E)\Omega_-(E)} \equiv e_p(\sigma)u^{-v_p(N_\sigma)}\frac{P_p(\sigma,u^{-1})}{P_p(\sigma,w^{-1})}\frac{L_{\{p,q|m\}}(E\otimes\sigma,1)}{\Omega_+(E)\Omega_-(E)} \mod p$$

*where $\sigma = 1\oplus\epsilon_p$ with $\epsilon_p$ the non-trivial character of $\mathbb{Q}(\mu_p)/\mathbb{Q}$ and $u, w$ such that,*

$$1 - a_pX + pX^2 = (1-uX)(1-wX), \quad u\in\mathbb{Z}_p^\times \ \ and \ \ p+1-a_p = \#E_p(\mathbb{F}_p)$$

Let us remark here that it is easy to see that we could relax our assumption that the conductor of $E$ equals the conductor of the mod $p$ representation in the expense of obtaining the weaker congruences,

$$R(\rho)\prod_{q|N_{diff}}P_q(E,\rho,1) \equiv R(\sigma)\prod_{q|N_{diff}}P_q(E,\sigma,1) \mod p$$

where $N_{diff}$ the ratio of the conductor of $E$ over the Artin conductor of the mod $p$ representation. Indeed, instead of considering the eigenform $f_0|\imath_m$ we need to consider the one where we remove the primes that divide $m$ and those that divide $N_{diff}$, i.e. $f_0|\imath_{mN_{diff}}$. It is this eigenform that will induce a homomorphism of the Hecke algebra that factors through a reduced local ring in case that $N_{diff}$ is not one. Then everything carries as above but eventually we remove also the Euler factors at $N_{diff}$ as we have modified $f$ in this way.

# 7 Speculations for the case $p > 3$

As the title indicates there are no real results in this section. The aim is to give a brief account of the problems that we face trying to extend our previous results to the case $p > 3$, working in the Hilbert modular form setting.

Note that in the previous section the fundamental result of Wiles allowed us to compare the size of the module of congruences for the Hecke algebra with the cohomological one which eventually was related to the periods that we used. In particular the crucial results were, first, that the localized first cohomology group was a free Hecke module over the local ring corresponding to our cusp form and second the generalization of "Ihara's lemma" that allowed us to relate the different levels. In the Hilbert modular form setting, results of this form have be obtained by Diamond in [9] and Dimitrov [10]. However for both authors it is crucial to assume that they work with a prime that is unramified in the totally real field.

Moreover there is another difficulty that is related to the automorphic periods that we can also define in this Hilbert modular forms setting. Indeed using the Eichler-Shimura-Harder isomorphism we can define periods $\Omega(\phi)_\Sigma$, the analogue of $\Omega(f)_\Sigma$, and their relation to the Petersson inner product is governed by the cohomological module of congruences. However even if we had an Ihara type lemma in this case we would have still to relate $\Omega(\phi)$, the minimal level, to the Néron periods up to $p$-adic units. So we run again into the same question as the one we addressed in our work [3], that is to understand the behavior of the automorphic periods under base change.

Having stated these problems we would like to speculate a little. Note that in what we said above we do not really make use of the fact that actually we consider a Hilbert modular form that is coming from base-change. In what follows we will try to indicate that perhaps one can avoid working over the totally real field and reduce our questions to the study of the adjoint square $L(ad(f), s)$ $L$-function associated to $f$ and its behavior under twists over the extension $F/\mathbb{Q}$, $F = \mathbb{Q}(\mu_p)^+$. This also will justify our choice to underline the identification of the local ring $h_2(\Gamma_0(N_\Sigma), \mathbb{Z}_p)_\mathfrak{m}$ with the "deformation" ring $T_\Sigma$, in the previous section. Our exposition is very brief and not rigorous.

So we keep the same notation as in the previous sections with the obvious extensions to the Hilbert modular case. That is $h$ is now a local ring of the Hecke algebra acting on the space of Hilbert cusp forms of level $N\mathfrak{p}m^2$ completed at $p$. Moreover it is the reduced local ring through which our ordinary normalized cusp form $\tilde{\phi}_0$ factors. We write $C_0(h)$ for its module of congruences. As in the elliptic case one can identify $h$ with the "deformation" ring $\mathbb{T}_\Sigma$. We consider the $\mathbb{Z}_{(p)}$-

module, $C_1(h) := (ker\pi_\Sigma)/(ker\pi_\Sigma)^2$. Then by [6] page 117, we have the inequality $|C_1(h)| \geq |C_0(h)|$. Let us now write $\rho_F$ for the 3-adic representation obtained from $\rho$ by restriction to $G_F$ and consider its reduction $\bar{\rho}_F$ modulo 3. Now let us impose the following conditions on $\rho_F$,

1. $\bar{\rho}_F$ is absolutely irreducible,
2. ($p$-ordinary) $\rho_F|_{D_\mathfrak{p}} \cong \begin{pmatrix} \delta_\mathfrak{p} & * \\ 0 & \epsilon_\mathfrak{p} \end{pmatrix}$ where $\delta_\mathfrak{p}$ is an unramified character.

Then it is known by the work of Mazur that there exists a universal deformation couple $(R_F, \varrho_F)$, in the terminology of Hida [17], that represents deformations with prescribed determinant and ramification in a way that we do not make explicit here. As we indicated in the elliptic case, the algebra $\mathbb{T}_\Sigma$ can be interpreted as a deformation algebra for $\bar{\rho}_F$ and hence there is a surjection $R_F \twoheadrightarrow \mathbb{T}_\Sigma$. This implies the inequality [6] p.118, $|C_1(R_F)| \geq |C_1(\mathbb{T}_\Sigma)| \geq |C_0(\mathbb{T}_\Sigma)|$. Again by Mazur's theory one can identify $C_1(R_F)$ with the Pontryagin dual of a properly defined Selmer group $Sel(ad(\rho_F))$ attached to $ad(\rho_F)$. But we can decompose $Sel(ad(\rho_F)) = \oplus_\chi Sel(ad(\rho) \otimes \chi)$ for $\chi \in Gal(F/\mathbb{Q})^V$. So back to our congruences we have a bound for our constant $c(\phi, m)$ by the the sizes of the Selmer groups of $ad(\rho)$ twisted by characters $\chi$ that factor through $Gal(F/\mathbb{Q})$.

Recall that the periods that appear in our congruences involve the Petersson inner product $< \tilde{\phi}_0|\tau_{N_\Sigma}, \tilde{\phi}_0 >$. We would like to factor this quantity to quantities that are related with the cusp form $f$ and more important we would like to obtain some control of the constants that may appear. In the elliptic modular forms case a formula of Shimura allows one to relate the Petersson inner product $< \tilde{f}_0|\tau_{N_\Sigma}, \tilde{f}_0 >$ to the value of the adjoint $L$ function $L(ad(f), s)$ at $s = 2$, in particular they are equal up to powers of $\pi$ and modified Euler factors at primes dividing $N_\Sigma$. This formula can be extended to the Hilbert modular case, see [27] page 669. One can then use the inductive properties of the $L$ functions to rewrite the periods $< \tilde{\phi}_0|\tau_{N_\Sigma}, \tilde{\phi}_0 >$ as a product of the form $\prod_\chi L(ad(f) \otimes \chi, 2)$, up to modified Euler factors and powers of $\pi$. These modified Euler factors should correspond to the local conditions that we have impose to the above mentioned Selmer group depending on the deformation problem.

Granted all the above speculations, we see that proving the congruences for $p > 3$ is closely related to the Tamagawa number conjecture for the adjoint square $L$ function of $f$ and its twists with characters over the extension $Gal(F/\mathbb{Q})$.

# References

[1] P.Balister, *Congruences Between Special Values of L-functions*, unpublished 1998

[2] Th.Bouganis *$L$-functions of elliptic curves and false Tate curve extensions*, Ph.D Thesis, Cambridge University, UK 2006

[3] Th.Bouganis, *Special values of L-functions and false Tate curve extensions I*, preprint.

[4] Th.Bouganis, V.Dokchitser, *Algebraicity of L-values for elliptic curves in a false Tate curve tower*, Math. Proc. of Cambridge Phil. Soc. Vol 142 March 2007.

[5] J.Coates, T.Fukaya, K.Kato, R.Sujatha, O.Venjakob, *The $GL_2$ main conjecture for elliptic curves without complex multiplication*, Publ. Math. IHES 101, 2005, 163-208

[6] H.Darmon, F.Diamond and R.Taylor, *Fermat's Last Theorem*, In Elliptic Curves, Modular Forms and Fermat's Last Theorem, ed. J.Coates and S.-T.Yau, series in Number theory I, International Press, Boston 1995

[7] D.Delbourgo and T.Ward, *Non abelian congruences between $L$-values of elliptic curves*, preprint (2006)

[8] P.Deligne and M.Rapoport, *Les schémas de modules des coubres elliptiques*, In:Modular functions of one variable II, Lecture notes in Math., vol 349, pp.143-316, Springer 1973

[9] F.Diamond, *The Taylor-Wiles construction and multiplicity one*, Inventiones Math. 128, 379-391 (1997)

[10] M.Dimitrov, *Galois representations modulo $p$ and cohomology of Hilbert modular varieties*, Ann. Sci. École Norm. Sup. (4) 38, 2005

[11] T.Dokchitser and V.Dokchitser *Computations in non-commutative Iwasawa theory (with an appendix by J.Coates and R.Sujatha)* Proc. London Math. Soc (3) 94 (2006) 211-272

[12] H.Hida, *Congruences of cusp forms and special values of their zeta functions*, Inventiones Math. 63 (1981), 225-261.

[13] H.Hida, *A p-adic measure attached to the zeta functions associated with elliptic modular forms I*, Inventiones Math. 79(1985), 159-195.

[14] H.Hida, *A p-adic measure attached to zeta functions associated with two elliptic modular forms II*, Ann. Inst. Fourier, Grenoble 38, 1-83(1988)

[15] H.Hida, *On p-adic L-functions of $GL_2 \times GL_2$ over totally real fields*, Ann. L' Institut Fourier, 41(1991), 311-391.

[16] H.Hida, *Elementary Theory of L-functions and Eisenstein Series*, Cambridge University Press, Cambridge 1993.

[17] H.Hida, *Modular Forms and Galois Cohomology*, Cambridge University Press, Cambridge 2000.

[18] K.Kato, *$K_1$ of some non-commutative completed group rings*, K-Theory 2005 34;99-140

[19] K.Kato, *p-adic Hodge Theory and values of zeta functions of modular forms*. Asterisque (2005), no.295

[20] N.Katz, *p-adic properties of modular schemes and modular forms*, International Summer School on Modular Functions, Anywerp 1972

[21] N.Katz, *p-adic interpolation of real analytic Eisenstein series*, Annals of Mathematics, 104(1976), 459-571

[22] N.Katz, *$p$-adic L-functions for CM fields*, Inventiones Math. 49, 199-297(1978)

[23] R. Langlands, *Base Change for $GL(2)$*, Annals of Math. Studies 96, Princeton University Press, 1980.

[24] B.Mazur, *Modular curves and the Eisenstein ideal*, Publ. Math. IHES 47 (1977)

[25] B.Mazur, *Rational isogenies of prime degree*, Inventiones Math. 44, 129-162(1978)

[26] B.Perrin-Riou, *Fonctions L p-adiques associees a une forme modulaire et a un corps quadratique imaginaire*, J. London Math. Soc. (2), 38, 1-32(1988)

[27] G.Shimura, *The Special Values of the Zeta Functions Associated with Hilbert Modular Forms*, Duke Mathematical Journal, vol 45,no 3 (1978)

[28] A.Wiles, *Modular elliptic curves and Fermat's Last Theorem*, Annals of Math. 141, 443-551(1995)